Continuation of

cut: R matrix talk
Recall: \( g \rightarrow \Phi \rightarrow \prod = \alpha_1 \cdots \alpha_k \) 

Let weight lattice \( d = [\Lambda = 2\Phi] \) 

\( U = U_q(G) \) 

\( U^+ = k \langle S \Phi \rangle \quad U^- = k \langle S \Phi \rangle \) 

\( (, ) : U^- \times U^+ \rightarrow k \) 

non-degenerate 

\( v_i^u \in U^- \quad u_i^+ \in U^+ \) 

\( \Theta_u = \sum v_i^u \otimes u_i^+ \) 

For \( w, v \in U - \text{mod} \) type \( r \) \( \Theta w, v = \sum_{u \geq 0} \Theta_u w / w_v \)
\[ f : W \otimes V \rightarrow W \otimes V \]
\[ \omega \otimes v \mapsto f \cdot \omega \otimes v \]
\[ \omega \otimes v \mapsto f \cdot \omega \otimes v \]

\[ R_{v,w} = \Theta_{w,v} \circ f \circ p \]

\[ R_{v,w} : \nu \otimes w \rightarrow w \otimes V \]

**Thm**

\( R_{v,w} \) is a \( \nu \)-nodule isomorphism, functorial in \( \nu \) \& \( W \).

\[ \nabla \uparrow \downarrow \]

\( \Delta(u) \circ \Theta = \Theta \circ \Delta'(u) \)

\( \Delta(u) \circ f = F \circ \Delta \circ \rho'(u) \)

\[ \tau \Delta = \tau \otimes \tau \Delta \]

\[ \tau (k) = K^{-1} \]
exercise $P(u \cdot x) = \Delta^0(u) P(x)$

exercise (use 0, 0, and 3) $R_v w (u \cdot x) = u \cdot R_v w (x)$

Proof of ① 
- Suffices to check on generators of $U$
  - $\Theta$ is in $(U \otimes U)$. 
    \[ \Delta(Ka) = Ka \otimes Ka = \Delta(Ka) \]
    implies $\Theta \circ \Delta(Ka) = \Delta(Ka) \circ \Theta$

$\Theta = \sum \Theta_n$
- $n = E\alpha$ and $n = F\alpha$ similar
  so we focus on $E\alpha$’s.

$E\alpha \otimes 1 \Theta u + (K\alpha \otimes E\alpha) \Theta u - \alpha = \Theta u (E\alpha \otimes 1) + \Theta u - q (K\alpha \otimes E\alpha)$
Start w/ \[ \theta u = \sum c_i u_i \]
\[ (E_\alpha \otimes 1) \circ \theta u - \theta \xi \circ (E_\alpha \otimes 1) = \sum (E_\alpha v_i^\mu - v_i^\mu E_\alpha) u_i^\mu \]

Jantzen's lemma 6.17 says for \( y \in U - u \), \( \alpha \in \Pi \), and \( u + \mathbb{Z} \frac{1}{\Gamma} \)
\[ E_\alpha y - y E_\alpha = c_\alpha (K_\alpha r_\alpha y) - r_\alpha'(y) K_\alpha^{-1} \]

Recall: \[ c_\alpha = (q_\alpha - q_\alpha^{-1})^{-1} = \frac{-1}{q_\alpha - q_\alpha^{-1}} \]
\[ y \in U_{u-x}, \quad \Delta(y) \in \bigoplus_{v \in U_{u-x}} U_{v-x} \]

so there are

\[ r_{x'}(y), \quad r_{x'}'(y) \in U_{u-x} \]

s.t.

\[ \Delta(y) = y \otimes K_u^{-1} + \sum_{\alpha \in \Pi} r_{x}(y) \otimes E_{\alpha} K_{u-\alpha}^{-1} + \text{(rest)} \]

\[ \Delta(y) = 1 \otimes y + \sum_{\alpha \in \Pi} F_{\alpha} \otimes r_{x'}(y) K_{\alpha}^{-1} + \text{(rest)} \]

So

\[ (E_1 \otimes 1) \Theta u - \Theta u (E_1 \otimes 1) = \sum_{\alpha} \sum_{i} K_{\alpha} r_{\alpha} (v^{i'}) - r_{\alpha'}(v^{i'}) K_{\alpha}^{-1} \otimes v^{i'} \]

\[ y \in U_{-(u-x)} \Rightarrow y = \sum (y, u^{i}_{j}-x) v^{j}_{i-x} \]
\[
= c_\alpha \sum_i (K_\alpha)_{ij} (\alpha(v_i^u, y_j^{u-1})) y_j^{u-2} \\
- \sum_i (\alpha'(v_i^u, y_j^{u-1})) y_j^{u-1} K_\alpha^{-1} x_i^u
\]

Tautzen (6.15)
\[
(\alpha(x, y)) = \frac{1}{c_\alpha} (y, E_\alpha x)
\]
\[
(\alpha'(x, y)) = \frac{1}{c_\alpha} (y, x E_\alpha)
\]

left as an exercise.

**Thm 2:** \( R_{(\cdot, \cdot)} \) satisfies the hexagon equations.
Proof: Let $m, m', m'' \in \mathcal{U}\text{-mod}$

want to show

$$m \otimes (m'^{\otimes m''}) \rightarrow (m \otimes m'') \otimes m'$$

$$m \otimes (m'^{\otimes m''})$$

commutes

or $R_{m \otimes m', m''} = (R_{m, m''} \otimes \text{id}) \circ (\text{id} \otimes R_{m', m''})$
\begin{align*}
\tag{1} \quad \Theta &= \sum q_i \otimes b_i \quad \Theta_{13} = \sum q_i \otimes 1 \otimes b_i \\
&= (P \otimes 1) \circ (1 \otimes \Theta) \circ (P \otimes 1) \\
\tag{2} \quad \tilde{f}_{13} (m \otimes m' \otimes m'') := f(\lambda_1, \lambda_3) m \otimes m' \otimes m'' \\
&= m_1 \otimes m_{\lambda_2} \otimes M_{\lambda_3} \\
&= (P \otimes 1) \circ (1 \otimes \tilde{f}) \circ (P \otimes 1) \\
\tag{3} \quad \Theta' &= \sum \frac{1}{r} (\Theta_{\nu})_{13} \circ (1 \otimes \kappa_{\nu} \eta_1)
\end{align*}
\[
\text{can check } \theta' \circ (f \otimes 1) = (f \otimes 1) \circ \theta_1 \\
\text{use Top of diagram } = (\theta \otimes 1) \circ \theta' \circ (f \otimes 1) \sim f_{13} \circ (P \otimes 1) \circ (1 \otimes P)
\]

\underline{Bottom of diagram}

\[
R_m \otimes n'' = \Theta m'' \circ P \circ (m \otimes m'), m''
\]

\[
(f \otimes 1) \circ f_{13} = f \left( \lambda_3, \lambda_1 + \lambda_2 \right) = f(\lambda_3, \lambda_1) \circ f(\lambda_3, \lambda_1)
\]

\[
\Theta m'', m \otimes m' = (\Theta \otimes \Delta)(\Theta)
\text{ check } = (\Theta \otimes 1) \circ \Theta'
\]
To "check"

\[ (\otimes \Delta)(\Theta u) = \sum_{0 \leq n < \eta} (\Theta_{u-\varnothing}) \circ (1 \otimes \nu \otimes u) \circ (\Theta)_{\nu} \]

use \( \Delta(x) = \sum (v_i \mapsto v_{ij}, x) \otimes v_{ij} \]

\( \left( \text{for } x \in \mathcal{U} \mathcal{u}^+ \right) \)

This verification is also left as an exercise.

use dual basis and that pairing is Hopf

\[ R = \Theta \circ f \circ (\otimes): V \otimes W \rightarrow W \otimes V \]

\[ \square \]
§3 Hecke Algebras and $q$–Schur–Weyl duality

- Schur–Weyl duality

$$\mathfrak{sl}_n \subset (\mathbb{C}^n)^{\otimes d} \hookrightarrow S_d^{\text{perm}}$$

permuting tensor factors

commuting actions $\Delta = \Delta^\circ$ for $U(\mathfrak{sl}_n)$

Theorem: $C[S_d] \to \text{End}_{\mathfrak{sl}_n}(\mathbb{C}^n^{\otimes d})$

(plus some description of kernel)

Definition: $H_{d,t} := \mathbb{Z}[t^{\pm 1}] Br_d/\langle \omega_i^2 = (t^i - t)e_i + 1 \rangle$
is the Hecke algebra.

- \( H_{d,t}=1 \cong \mathbb{Z}[S_d] \)
- \( H_{d,t} \) is a free \( \mathbb{Z}[t^{\pm 1}] \) module of rank \( d! \)

quantum Schur Weyl

\[
U = U_q(\mathfrak{sl}_n) \\
V = \mathbb{C}(\Xi_1)
\]

\[
R_i : V \otimes d \to V \otimes d \\
R_i = \text{id} \otimes R_{\Delta} \otimes \text{id}
\]

\[
\overset{L_i \text{ of terms}}{\overbrace{L_i \text{ of terms}}}
\]

- \( R_i R_{i+1} R_i = R_{i+1} R_i R_i \) ✓
- \( R_i R_j = R_j R_i \) \( |i-j| \geq 1 \) ✓
for quadratic relation, we look at example of $2l_2$

$$R_{v,v} = \begin{pmatrix}
0 & q^{1/2} & 0 & q^{-1/2} \\
q^{1/2} & q^{-1/2} & 0 & q^{-3/2} \\
0 & 0 & q & 0 \\
q^{-1/2} & q^{3/2} & 0 & q^{-1}
\end{pmatrix}$$

exercise

$$(R - q^{-1/2})(R + q^{3/2}) = 0$$

so

$$(q^{-1/2}R - q^{-1})(q^{-1/2}R + q) = 0$$

Let $t = q$

$$R_{Br_d} \xrightarrow{\epsilon_i} q^{-1/2}R_i$$
Thm (Jimbo) \( q \) not a root of unity, up to order \( n \)

\[
\mathbb{H}_{d,t} \otimes R \rightarrow \text{End}_U(V \otimes d)
\]

(plus some description of kernel, \( \text{id} \leq n \) then injective)

- \( q \) transcendental, the result follows from classical case
- result still true when \( q \) is a root of unity
  - when \( U \) is “enlarged” to divided powers form of \( U \)

exercise when \( q = a^{1/3} \), use \( \Theta_2, \Theta_3, \Theta_4 + \beta \)

check \( (q^{-1/3} R^2) = (q^{-1} - q)(q^{1/3} R) + 1 \)
The quantum coordinate algebra, $\mathbb{H}_q[G]$. 

$\mathfrak{g} \mapsto G$ connected, simply connected, semisimple algebraic group $\mathbb{K}$

- $k[G] = \text{ring of regular functions on } G$
- $\text{Der } G = \{ X \in \text{End}_{\mathbb{K} G} ; \ X(fg) = X(tf) + tX(g) \}$
- $L(G) = \{ X \in \text{Der } G ; \ X \circ L_g = L_{g \circ X} \}$

- $D_e(G) = \text{subalgebra of } \text{End}_{\mathbb{K} G}$
  - generated by $L(G)$
  - left invariant differential operators

since $\mathfrak{g} \cong L(G)$

$U(\mathfrak{g}) \longrightarrow D_e(G)$
Thm (Cartier) since char(\text{rk}) = 0
\[ U(G) \xrightarrow{\sim} D_1(G) \]

- Get a pairing \[ U(G) \times \mathbb{K}[G] \rightarrow \mathbb{R} \]
  \[ \langle X, f \rangle = X(f)(1) \]

Thm \[ \langle \cdot, \cdot \rangle \] induces an embedding

\[ \mathbb{K}[G] \xrightarrow{-} U(G)^* \]
\[ f \mapsto \langle \cdot, f \rangle \]

Matrix coefficients

\[ V \in U(G) - \text{mod} \]
\[ V \in V^* \quad f \in V^* \]

Define \[ c_{f, V} \in U(G)^* \] by
\[ c_{f, V}(X) = f(Xv) \]
Convolution product on $U(\log)^*$

$\lambda_1, \lambda_2 \in U(\mathfrak{g})^*$

$\lambda_1 \lambda_2 (u) = \exists \lambda_1 (u \cdot) \lambda_2 (u \cdot)$

for $u \in \mathfrak{u}$ s.t. $\Delta (u) = \exists u \cdot \Delta u \cdot$

Lemma

1. $C_{f, v} \cdot C_g, w = C_{f \circ g, v \circ w}$

   for $v \in \mathfrak{v}, f \in \mathfrak{v}^*, w \in \mathfrak{w}, g \in \mathfrak{w}^*$

2. $C_{f, v} + C_g, w = C_{f \circ g, (v, w)}$

3. $\varepsilon \in U(\mathfrak{g})^*$ and $\varepsilon \cdot \lambda = \lambda$

   $\varepsilon = C_{1^* 1^*} 1 \in \mathbb{R}$

[trivial module]
Proof exercise.

Main Point \[ \text{im } ( \mathbb{K}[G] \hookrightarrow U(g)^* ) \]

is a unital algebra (w/r/t convolution) on \( U(g)^* \)
spanned by matrix coeffs of all finite dim'd \( U(g) \)-modules.

To define \( \mathbb{K}_g[G] \) there is no \( G \), but we have

\[ \text{Defn } \mathbb{K}_g[G] = \text{unital subalgebra of } (U(g)^*)^* \]
spanned by matrix coeffs of all modules in \( U(g) \)-mod^{type 1} \]
Upshot:
\[ \chi : \mathbb{N} \to \mathbb{U} \]
\[ \chi : \mathbb{U} \to (\mathbb{N}^{\mathbb{U}})^{\mathbb{U}} \]

Lemma:
\[ \chi^* / : \mathbb{K}q[G] \to \mathbb{K}q[G] \otimes \mathbb{K}q[G] \]

\[ \mathbb{K}q[G] \]

so we can define comultiplication for \( \mathbb{K}q[G] \)

There is also a counit and antipode
so \( \mathbb{K}q[G] \) is a Hopf algebra

Proof exercise.
What is this algebra?

We can determine some relations from \( R \)

Let \( V, W \in U \)-mod type \( I \)

\[
\begin{align*}
V_j & \rightarrow V_n & W_j & \rightarrow W_m & c_{ij} = c_{V_j V_n} \\
V^*_j & \rightarrow V^*_n & W^*_j & \rightarrow W^*_m & d_{ij} = d_{W^*_j W^*_m}
\end{align*}
\]

\[
R_{w,v} : W \otimes V \xrightarrow{\sim} V \otimes W
\]

\[
\omega \otimes v_j \mapsto \sum_{h,l} R_{ij}^h v_n \otimes w_e
\]

Lemma: The following relations hold in \( \mathcal{L} \)

\[
\sum_{h,e} R_{ne} d_{li} c_{ij} = \sum_{h,l} R_{ij}^h c_{li} d_{ne} e
\]
Proof since

\[ C \omega^+ \otimes v^+ \omega \otimes v_j = C \omega^+ \omega_i \cdot C v^+ v_j = d_{i} c_{i j} \]

\[ u \cdot w_i \otimes v_j = \sum_{j \neq l} d_{i} c_{i j} (u) w_l \otimes v_l \]

Expand both sides of

\[ R ( u \cdot w_i \otimes v_j ) = u \cdot ( R ( w_i \otimes v_j ) ) \]

and compare coefficients.
example \[ H_q \left[ SL_2 \right] \]

\[ V = W = L(\omega_1) \]

\[ v_1 = v \quad v_2 = Fv \]

exercise use \( R_{v,v} \) to deduce

\[ c_{11} c_{12} = q c_{12} c_{11} \quad c_{11} c_{21} = q c_{21} c_{11} \]

\[ c_{12} c_{22} = q c_{22} c_{12} \quad c_{21} c_{22} = q c_{22} c_{21} \]

\[ c_{12} c_{21} = c_{21} c_{12} \]

\[ c_{11} c_{22} - c_{22} c_{11} = (q^{-1} - q) c_{12} c_{21} \]
Facts

- $c_1^g$'s generate $\mathbb{R}_q[SL_2]$ as an algebra (Lenard condition)

- adding

$$c_{11}c_{22} - q \cdot c_{12}c_{21} = 1$$

gives a complete list of relations