Continuation of

Ch 7: R matrix talk

Recall: $\mathfrak{g} \rightsquigarrow \Phi \supset \Pi = \{\alpha_1 \dots \alpha_\ell\}$

$\mathbb{k}$ field $\quad \Lambda$ weight lattice $\quad d = [\Lambda : \mathbb{Z}\Phi]$

$q \in \mathbb{k}$ $\quad \mathcal{U} = \mathcal{U}_q(\mathfrak{g})$

$\operatorname{char}(\mathbb{k}) = 0$

$q^{1/d} \in \mathbb{k}$ $\quad \mathcal{U}^+ = \mathbb{k}\langle E_\alpha \rangle \quad \mathcal{U}^- = \mathbb{k}\langle F_\alpha \rangle$

$(\ , \ ) : \mathcal{U}^- \times \mathcal{U}^+ \longrightarrow \mathbb{k}$

non degenerate

$v_i^\mu \in \mathcal{U}^-_\mu \quad u_i^\mu \in \mathcal{U}^+_\mu$

$\Theta_\mu = \sum v_i^\mu \otimes u_i^\mu$

For $W, V \in \mathcal{U}\text{-mod}^{\text{type } 1}$ • $\Theta_{W,V} = \sum_{\mu \geq 0} \Theta_\mu / W \otimes V$

- $\tilde{f} : W \otimes V \longrightarrow W \otimes V$

$$w \otimes v \longmapsto q^{-(\lambda,\mu)} \cdot w \otimes v$$

$$\begin{array}{c} \cap \\ W_{\lambda_1} \otimes V_{\lambda_2} \end{array}$$

- $R_{V,W} = \Theta_{W,V} \circ \tilde{f} \circ P$

$R_{V,W} : V \otimes W \to W \otimes V$

Thm $R_{V,W}$ is a $\mathcal{U}$-module isomorphism, functorial in V & W.

NTS ① $\Delta(u) \circ \Theta = \Theta \circ {}^{\tau}\Delta(u)$

$${}^{\tau}\Delta = \tau \otimes \tau \circ \Delta \circ \tau$$

$$\tau(K) = K^{-1}$$

② ${}^{\tau}\Delta(u) \circ \tilde{f} = \tilde{f} \circ \Delta^{op}(u)$

exercise ③ $P(u\cdot x) = \Delta^{op}(u)\, P(x)$

exercise (use ①, ②, and ③) $R_{V,W}(u\cdot x) = u \cdot R_{V,W}(x)$

Proof of ①

- Suffices to check on generators of $\mathcal{U}$
- Θ is in $(\mathcal{U}\otimes\mathcal{U})_0$

$\Delta(K_\alpha) = K_\alpha \otimes K_\alpha = {}^{t}\Delta(K_\alpha)$

implies $\Theta \circ {}^{\tau}\Delta(K_\alpha) = \Delta(K_\alpha)\circ\Theta$

- $u = E_\alpha$ and $u = F_\alpha$ similar so we focus on E_α's.

$\Theta = \sum \Theta_\eta$

suffices to show

$$(E_\alpha\otimes 1)\,\Theta_\eta + (K_\alpha\otimes E_\alpha)\,\Theta_{\eta-\alpha} = \Theta_\eta\,(E_\alpha\otimes 1) + \Theta_{\eta-\alpha}\,(K_\alpha^{-1}\otimes E_\alpha)$$

$\Delta(E_\alpha)$ $\qquad\qquad$ ${}^{\tau}\Delta(E_\alpha)$

- Start w/ $\qquad \Theta_\mu = \sum r_i^{-\mu} \otimes x_i^{-\mu}$

$$(E_\alpha \otimes 1) \circ \Theta_\mu - \Theta_\mu \circ (E_\alpha \otimes 1) = \sum (E_\alpha v_i^{-\mu} - v_i^{-\mu} E_\alpha) \otimes u_i^{\mu}$$

- Jantzen's lemma 6.17 says for $y \in U^-_{-\mu}$, $\alpha \in \Pi$, and $\mu \in \mathbb{Z}\Phi$

$$E_\alpha y - y E_\alpha = c_\alpha \left(K_\alpha r_\alpha(y) - r_\alpha'(y) K_\alpha^{-1} \right)$$

<u>recall</u>: $\quad$ • $c_\alpha = (q_\alpha - q_\alpha^{-1})^{-1} = \dfrac{-1}{q_\alpha - q_\alpha^{-1}}$

- $y \in U^-_{-\mu}$, $\Delta(y) \in \bigoplus_{0 \leq \nu \leq \mu} U^-_{-\nu} \otimes U^-_{-(\mu-\nu)} K_\nu^{-1}$

so there are
$r_\alpha(y), r'_\alpha(y) \in U^-_{-(\mu-\alpha)}$ s.t.

Def'n from 6.15

- $\Delta(y) = y \otimes K_\mu^{-1} + \sum_{\alpha \in \Pi} r_\alpha(y) \otimes E_\alpha K_{\mu-\alpha}^{-1} + (\text{rest})$
- $\Delta(y) = 1 \otimes y + \sum_{\alpha \in \Pi} F_\alpha \otimes r'_\alpha(y) K_\alpha^{-1} + (\text{rest})$

So

$$(E_\alpha \otimes 1)\Theta_\mu - \Theta_\mu(E_\alpha \otimes 1) = c_\alpha \sum \left(K_\alpha r_\alpha(v_i^\mu) - r'_\alpha(v_i^\mu) K_\alpha^{-1}\right) \otimes u_i^\mu$$

$$y \in U_{-(\mu-\alpha)} \Rightarrow y = \sum (y, u_j^{\mu-\alpha})\, v_j^{\mu-\alpha}$$

$$= c_\alpha \sum_i \Big(K_\alpha \sum_j (r_\alpha(v_i^{-\mu}), u_j^{\mu-\alpha}) v_j^{\mu-\alpha}$$

$$- \sum_j (r_\alpha'(v_i^{-\mu}), u_j^{\mu-\alpha}) v_j^{\mu-\alpha} K_\alpha^{-1} \Big) \otimes u_i^{\mu}$$

Jantzen (6.15)

$$(r_\alpha(y), x) = \frac{1}{c_\alpha}(y, E_\alpha x)$$

$$(r_\alpha'(y), x) = \frac{1}{c_\alpha}(y, x E_\alpha)$$

rest is left as an exercise. $\square$

Thm 2 $R_{(\ ),(\ )}$ satisfies the hexagon equations

Proof Let $M, M', M'' \in \mathcal{U}\text{-mod}^{\text{type } 1}$

want to show

$$\begin{tikzcd}
 & M\otimes(M''\otimes M') \arrow[r] & (M\otimes M'')\otimes M' \arrow[dr] & \\
M\otimes(M'\otimes M'') \arrow[ur] \arrow[dr] & & & (M''\otimes M)\otimes M' \\
 & (M\otimes M')\otimes M'' \arrow[r] & M''\otimes(M\otimes M') \arrow[ur] &
\end{tikzcd}$$

commutes

or $R_{M\otimes M', M''} = (R_{M,M''}\otimes \mathrm{id})\circ(\mathrm{id}\otimes R_{M',M''})$

Top of diagram

$$(\theta\otimes 1)\circ(\tilde f\otimes 1)\circ(P\otimes 1)\circ(1\otimes\theta)\circ(1\otimes\tilde f)\circ(1\otimes P)$$

① $\theta = \sum a_i\otimes b_i \qquad \theta_{13} = \sum a_i\otimes 1\otimes b_i$

$$\theta_{13} = (P\otimes 1)\circ(1\otimes\theta)\circ(P\otimes 1)$$

② $\tilde f_{13}(m\otimes m'\otimes m'') := f(\lambda_1,\lambda_3)\, m\otimes m'\otimes m''$

$$M_{\lambda_1}\otimes M'_{\lambda_2}\otimes M''_{\lambda_3}$$

$$\tilde f_{13} = (P\otimes 1)\circ(1\otimes\tilde f)\circ(P\otimes 1)$$

③ $\theta' = \sum_\eta (\theta_\eta)_{13}\circ(1\otimes K_\eta\otimes 1)$

can check $\Theta' \circ (\tilde{f} \otimes 1) = (\tilde{f} \otimes 1) \circ \Theta_{13}$

use ① ② ③ Top of diagram $= (\Theta \otimes 1) \circ \Theta' \circ (\tilde{f} \otimes 1) \tilde{f}_{13} (P \otimes 1) \circ (1 \otimes P)$

Bottom of diagram

$$R_{M \otimes M', M''} = \Theta_{M'', M \otimes M'} \circ \tilde{f} \circ P_{(M \otimes M'), M''}$$

- $(\tilde{f} \otimes 1) \circ \tilde{f}_{13} = \tilde{f}$ $\quad f(\lambda_3, \lambda_1 + \lambda_2) = f(\lambda_3, \lambda_1) f(\lambda_3, \lambda_1)$

- $\Theta_{M'', M \otimes M'} = (1 \otimes \Delta)(\Theta)$

check
$= (\Theta \otimes 1) \circ \Theta'$

to "check"

verify $(1\otimes\Delta)(\Theta_\mu) = \sum_{0\le\nu\le\mu} \left(\Theta_{\mu-\nu}\otimes 1\right)\circ(1\otimes K_\nu\otimes 1)\circ(\Theta_\nu)_{13}$

use $\Delta(x) = \sum_{i,j,\nu} \left(v_i^{\mu-\nu} v_j^{\nu}, x\right) u_i^{\mu-\nu} K_\nu \otimes u_j^{\nu}$

(for $x\in U_\mu^+$)

This verification is also left as an exercise.

use dual basis and that pairing is Hopf $\square$

$R = \Theta\circ\tilde{f}\circ P : V\otimes W \to W\otimes V$

§3 Hecke Algebras and q-Schur-Weyl duality

- Schur Weyl duality

$$\mathfrak{gl}_n \circlearrowright (\mathbb{C}^n)^{\otimes d} \circlearrowleft S_d$$

permuting tensor factors

$$\underset{i,i+1\text{ st}}{\mathrm{id}\otimes P\otimes \mathrm{id}} \longleftarrow\!\shortmid s_i$$

commuting actions $\Delta = \Delta^{op}$ for $\mathcal{U}(\mathfrak{gl}_n)$

Theorem: $$\mathbb{C}[S_d] \twoheadrightarrow \mathrm{End}_{\mathfrak{gl}_n}((\mathbb{C}^n)^{\otimes d})$$

(plus some description of kernel)

Defn $$\mathbb{H}_{d,t} := \mathbb{Z}[t^{\pm 1}]\,Br_d \big/ \langle \sigma_i^2 = (t^{-1}-t)\sigma_i + 1 \rangle$$

is the <u>Hecke algebra</u>.

- $\mathbb{H}_{d,t=1} \cong \mathbb{Z}[S_d]$
- $\mathbb{H}_{d,t}$ is a free $\mathbb{Z}[t^{\pm 1}]$ module of rank $d!$

<u>quantum Schur Weyl</u>

$\mathcal{U} = \mathcal{U}_q(\mathfrak{sl}_n)$

$V = L(\varpi_1)$

$$R_i : V^{\otimes d} \longrightarrow V^{\otimes d}$$

$$R_i := \mathrm{id} \otimes \underbrace{R_{V,V}}_{i,\, i+1^{st} \text{ terms}} \otimes \mathrm{id}$$

- $R_i R_{i+1} R_i = R_{i+1} R_i R_{i+1}$ ✓
- $R_i R_j = R_j R_i \qquad |i-j| > 1$ ✓

$$\mathrm{Br}_d \longrightarrow \mathrm{End}_{\mathcal{U}}(V^{\otimes d})$$
$$\sigma_i \longmapsto R_i$$

for quadratic relation, we look at example of $\mathfrak{sl}_2$

$$R_{V,V} = \begin{pmatrix} q^{-1/2} & & & \\ & 0 & q^{1/2} & \\ & q^{1/2} & q^{-1/2} - q^{3/2} & \\ & & & q^{-1/2} \end{pmatrix}$$

exercise
$$(R - q^{-1/2})(R + q^{3/2}) = 0$$

so $$(q^{-1/2}R - q^{-1})(q^{-1/2}R + q) = 0$$

Let $t = q$

$$\sigma_i \mapsto q^{-1/2} R_i$$

$$\begin{tikzcd}
& \Bbbk\mathrm{Br}_d \arrow[dl] \arrow[dr] & \\
\Bbbk \otimes H_{d,t} \arrow[rr] & & \mathrm{End}_{\mathcal{U}}(V^{\otimes d})
\end{tikzcd}$$

$$\bar{\sigma}_i \longmapsto q^{-1/2} R_i$$

Thm (Jimbo) q not a root of unity, up to order n

$$\mathbb{R} \otimes \mathbb{H}_{d,t} \twoheadrightarrow \mathrm{End}_{\mathcal{U}}(V^{\otimes d})$$

(plus some description of kernel, if $d \leq n$ then injective)

- q transcendental, the result follows from classical case
- result still true when q is a root of unity when $\mathcal{U}$ is "enlarged" to divided powers form of $\mathcal{U}$

exercise when $\mathfrak{g} = \mathfrak{sl}_3$, use $\theta_\alpha, \theta_\beta, \theta_{\alpha+\beta}$

check $(q^{-1/3} R)^2 = (q^{-1} - q)(q^{-1/3} R) + 1$

§4 The quantum coordinate algebra, $\Bbbk_q[G]$.

$\mathfrak{g} \rightsquigarrow G$ connected, simply connected, semisimple algebraic group$/\Bbbk$

char$(\Bbbk)=0$

- $\Bbbk[G]$ = ring of regular functions on G
- $\mathrm{Der}(G) = \{ X \in \mathrm{End}_{\Bbbk}(\Bbbk[G]) \mid X(fg) = X(f)g + fX(g) \}$
- $L(G) = \{ X \in \mathrm{Der}(G) \mid X \circ l_g = l_g \circ X \ \ \forall g \in G \}$
- $D_e(G)$ = subalgebra of $\mathrm{End}_{\Bbbk}(\Bbbk[G])$ generated by $L(G)$

left invariant differential operators

since $\mathfrak{g} \cong L(G)$

$$\mathcal{U}(\mathfrak{g}) \longrightarrow D_e(G)$$

Thm (Cartier) since char($\Bbbk$) = 0

$$\mathcal{U}(\mathfrak{g}) \xrightarrow{\cong} D_e(G)$$

- Get a pairing $\mathcal{U}(\mathfrak{g}) \times \Bbbk[G] \longrightarrow \Bbbk$

$$\langle X, f \rangle = X(f)(I)$$

Thm $\langle\ ,\ \rangle$ induces an embedding

$$\Bbbk[G] \hookrightarrow \mathcal{U}(\mathfrak{g})^*$$
$$f \longmapsto \langle -, f \rangle$$

Matrix coefficients

$V \in \mathcal{U}(\mathfrak{g})$ - mod

$v \in V \quad f \in V^*$

define $c_{f,v} \in \mathcal{U}(\mathfrak{g})^*$ by $c_{f,v}(X) = f(Xv)$

Convolution product on $\mathcal{U}(\mathfrak{g})^*$

$\lambda_1, \lambda_2 \in \mathcal{U}(g)^*$

$$\lambda_1 \lambda_2 (u) = \sum \lambda_1(u_i) \lambda_2(u_i')$$

for $u \in \mathcal{U}$ s.t. $\Delta(u) = \sum u_i \otimes u_i'$

<u>lemma</u>

① $c_{f,v} \cdot c_{g,w} = c_{f \otimes g, v \otimes w}$

for $v \in V$, $f \in V^*$, $w \in W$, $g \in W^*$

② $c_{f,v} + c_{g,w} = c_{f \oplus g, (v,w)}$

③ $\varepsilon \in \mathcal{U}(g)^*$ and $\varepsilon \cdot \lambda = \lambda$

trivial module

$\varepsilon = c_{1^*, 1}$ $\quad 1 \in \mathbb{R}$

Proof exercise.

Main Point $\operatorname{im}\left(\mathbb{k}[G] \hookrightarrow U(\mathfrak{g})^*\right)$

is a unital algebra (w/r/t convolution on $U(\mathfrak{g})^*$)

spanned by matrix coeffs

of all finite dim'l $U(\mathfrak{g})$-modules

To define $\mathbb{k}_q[G]$ there is no G, but we

have

Defn $\mathbb{k}_q[G]$ = unital subalgebra of $(U_q(\mathfrak{g}))^*$
spanned by matrix coeffs
of all modules in $U(\mathfrak{g})\text{-mod}^{\text{type } 1}$

Upshot $\quad \mu : \mathcal{U} \otimes \mathcal{U} \longrightarrow \mathcal{U}$

$$\mu^* : \mathcal{U}^* \longrightarrow (\mathcal{U} \otimes \mathcal{U})^* \not\supseteq \mathcal{U}^* \otimes \mathcal{U}^*$$

lemma $\quad \mu^*\big|_{\Bbbk_q[G]} : \Bbbk_q[G] \longrightarrow \Bbbk_q[G] \otimes \Bbbk_q[G]$

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so we can define comultiplication for $\Bbbk_q[G]$

There is also a counit and antipode

so $\Bbbk_q[G]$ is a Hopf algebra

Proof exercise.

- What is this algebra?

We can determine some relations from R

- Let $V, W \in U\text{-mod}^{\text{type } 1}$

$v_1 \dots v_n \quad w_1 \dots w_m$

$v_1^* \dots v_n^* \quad w_1^* \dots w_n^*$

$c_{ij} := c_{v_i^*, v_j}$

$d_{ij} := d_{w_i^*, w_j}$

$$R_{W,V} : W \otimes V \xrightarrow{\sim} V \otimes W$$

$$w_i \otimes v_j \longmapsto \sum_{h,l} R_{ij}^{hl} \, v_h \otimes w_l$$

<u>lemma</u> The following relations hold in $\Bbbk_q[G]$

$$\sum_{h,l} R_{hl}^{rs} d_{li} c_{hj} = \sum_{h,l} R_{ij}^{hl} c_{rh} d_{sl}$$

Proof Since

$$C_{w_l^* \otimes v_n^*, w_i \otimes v_j} = C_{w_l^*, w_i} \cdot C_{v_n^*, v_j} = d_{li} c_{nj}$$

$$u \cdot w_i \otimes v_j = \sum_{n,l} d_{li}\, c_{nj}(u)\, w_l \otimes v_n$$

Expand both sides of

$$R(u \cdot w_i \otimes v_j) = u \cdot (R(w_i \otimes v_j))$$

and compare coefficients. $\square$

example $\mathbb{k}_q[SL_2]$ $V = W = L(\varpi_1)$

$v_1 = v$ $v_2 = Fv$

exercise use $R_{V,V}$ to deduce

$c_{11}c_{12} = q\, c_{12}c_{11}$ $c_{11}c_{21} = q\, c_{21}c_{11}$

$c_{12}c_{22} = q\, c_{22}c_{12}$ $c_{21}c_{22} = q\, c_{22}c_{21}$

$c_{12}c_{21} = c_{21}c_{12}$

$c_{11}c_{22} - c_{22}c_{11} = (q - q^{-1})\, c_{12}c_{21}$

Facts

- c_{ij}'s generate $\mathbb{k}_q[SL_2]$ as an algebra $\left(L(n\omega_1) \subset V^{\otimes n} \right)$

- adding

$$c_{11}c_{22} - q\, c_{12}c_{21} = 1$$

gives a complete list of relations