JANTZEN CH 7 - R MATRICES

CONTENTS

0. INTRODUCTION

Let $\Bbbk$ be a field with characteristic not equal to two, and let $q \in \Bbbk$ such that $q^n \neq 1$ for all n. Let V and W be two finite dimensional (type **1**) $U_q(\mathfrak{g})$ modules. Weight considerations tell us that $V \otimes W$ and $W \otimes V$ are isomorphic. However, the obvious vector space isomorphism $P : V \otimes W \xrightarrow{v\otimes w \mapsto w \otimes v} W \otimes V$ does not commute with the action of U. We aim to define isomorphisms $R_{V,W} : V \otimes W \longrightarrow W \otimes V$ which do commute with the action of U. Then we will show that the $R_{V,W}$ satisfy nice properties: functorality in V and W, the hexagon identity, and solution to quantum Yang-Baxter.

1. DEFINITION OF THE QUASI R MATRIX

1.1. **Motivation for quasi R matrix.** Let us suppose that there is a functorial isomorphism $R_{M,M'} : M \otimes M' \longrightarrow M' \otimes M$ for all representations of U (so not just finite dimensional ones). Then we may consider $R = R_{_UU,_UU}(1 \otimes 1)$. If M is a U-module and $m \in M$ then we get a homomorphism of U-modules

$$\rho_m : {}_UU \to M, \ u \mapsto um.$$

Let M and M' be two U modules and let $m \in M$ and $m' \in M'$. Then to compute $R_{M,M'}(m \otimes m')$ we see by functorality of $R_{(-),(-)}$ that

$$R_{M,M'}(m \otimes m') = R_{M,M'} \circ \rho_m \otimes \rho_{m'}(1 \otimes 1) = \rho_{m'} \otimes \rho_m \circ R_{_UU,_UU}(1 \otimes 1) = \rho_{m'} \otimes \rho_m(R) = R \cdot m' \otimes m$$

so $R_{M,M'}(m \otimes m') = RP(m \otimes m')$.

That $R_{(-),(-)}$ is a functorial *isomorphism* implies that R is invertible in $U \otimes U$ while $R_{(-),(-)}$ being a morphism of U modules means

$$\Delta(u)RP(m \otimes m') = u \cdot R_{M,M'}(m \otimes m') = R_{M,M'}(u \cdot m \otimes m') = RP(\Delta(u)m \otimes m').$$

If $\Delta(u) = \sum u_i \otimes u_i'$, then

$$\Delta(u)R(m' \otimes m) = \Delta(u)RP(m \otimes m') = RP(\Delta(u)m \otimes m') = R\sum u_i' m' \otimes u_i m = R\Delta^{op}(u)(m' \otimes m).$$

Hence,

$$\Delta(u) \cdot R = R \cdot \Delta^{op}(u),$$

where $\Delta^{op}(u) = P(\Delta(u))$. Note that $\Delta^{op} \neq \Delta$, or as we learned in previous lectures U is not co-commutative.

The element R was constructed by Drinfeld, but lies in a completion of U. It is an expression of the form

$$R = q^{-\sum h_\alpha \otimes h'_\alpha}(1 + \ldots),$$

Here the h_α and the h'_α are dual bases for $\mathfrak{h} \subset \mathfrak{g}$ and $q^{h_\alpha} = K_\alpha$. The ... term is a sum of tensors of dual basis elements $x \otimes y \in U^- \otimes U^+$.

We follow Jantzen, so only use this element as motivation for a construction of $\theta^f_{V,W}$ for V and W finite dimensional, type **1** representations of U.

1.2. **The $\mathfrak{sl}_2$ case.** Recall that $U = U_q(\mathfrak{sl}_2)$ is defined by generators and relations, with generators $E, F, K^{\pm 1}$. Furthermore, U is a Hopf algebra with coproduct Δ defined as the algebra homomorphism $U \longrightarrow U \otimes U$ defined on generators by

$$\Delta(E) = E \otimes 1 + K \otimes E, \tag{1.1}$$

$$\Delta(F) = F \otimes K^{-1} + 1 \otimes F, \tag{1.2}$$

and

$$\Delta(K) = K \otimes K. \tag{1.3}$$

In particular, when V and W are two representations of U we can produce a new representation $V \otimes W$, where $u \cdot (v \otimes w) = \Delta(u) \cdot (v \otimes w)$. In fact we can produce two new representations $V \otimes W$ and $W \otimes V$.

There is always a vector space isomorphism $P : V \otimes W \longrightarrow W \otimes V$ which sends a simple tensor $v \otimes w$ to $w \otimes v$. When $q = 1$, this map is an isomorphism of $\mathfrak{sl}_2$ modules, but this will not be so for generic q (this is motivation only, you don't actually recover classical $\mathfrak{sl}_2$ when $q = 1$). Instead, we hope to find isomorphisms $\theta^f_{V,W} : V \otimes W \longrightarrow V \otimes W$ so that $R_{V,W} := \theta^f_{V,W} \circ P$ is a U-module isomorphism $V \otimes W \xrightarrow{\sim} W \otimes V$. That is we want to "deform" the flip map P by some map θ^f so that the "deformed" flip map $\theta^f \circ P$ does commute with the action of U.

It turns out the answer starts with

$$\theta_n = (-1)^n q^{-\binom{n}{2}} \frac{(q - q^{-1})^n}{[n]!} F^n \otimes E^n. \tag{1.4}$$

and

$$\theta_{V,W} = \sum_{n \geq 0} \theta_n|_{V \otimes W}. \tag{1.5}$$

Both E and F act nilpotently in each finite dimensional representation of U, so we can choose a basis so that $E \otimes F$ is strictly upper triangular (b/c nilpotent). In this basis each

θ_n, $n \geq 1$, is strictly upper triangular. Since $\theta_0 = 1$ we deduce $\theta_{V,W}$ is an invertible (in fact unipotent) linear transformation.

Recall that there is an anti-automorphism of U, $\tau : U \to U$, defined on generators by

$$E \mapsto E, F \mapsto F, \text{ and } K \mapsto K^{-1}.$$

Given an anti-automorphism, one can twist Δ to obtain another comultiplication

$$ {}^{\tau}\Delta = \tau \otimes \tau \circ \Delta \circ \tau^{-1}. \tag{1.6}$$

Thus, ${}^{\tau}\Delta$ is defined on generators by

$$ {}^{\tau}\Delta(E) = E \otimes 1 + K^{-1} \otimes E \tag{1.7}$$

$$ {}^{\tau}\Delta(F) = F \otimes K + 1 \otimes F \tag{1.8}$$

$$ {}^{\tau}\Delta(K) = K \otimes K. \tag{1.9}$$

Lemma 1.1. *For all $u \in U$*

$$\Delta(u) \circ \theta_{V,W} = \theta_{V,W} \circ {}^{\tau}\Delta(u) \tag{1.10}$$

Proof. This follows from checking that for all $n \geq 0$

$$(E \otimes 1)\theta_n + (K \otimes E)\theta_{n-1} = \theta_n(E \otimes 1) + \theta_{n-1}(K^{-1} \otimes E) \tag{1.11}$$

$$(1 \otimes F)\theta_n + (F \otimes K^{-1})\theta_{n-1} = \theta_n(1 \otimes F) + \theta_{n-1}(F \otimes K) \tag{1.12}$$

$$(K \otimes K)\theta_n = \theta_n(K \otimes K). \tag{1.13}$$

The computation is left as an exercise. □

Remark 1.2. Recall that

$$\Delta^{op}(E) = 1 \otimes E + E \otimes K \tag{1.14}$$

$$\Delta^{op}(F) = K^{-1} \otimes F + F \otimes 1 \tag{1.15}$$

$$\Delta^{op}(K) = K \otimes K \tag{1.16}$$

so ${}^{\tau}\Delta \neq \Delta^{op}$.

We want to tweak $\theta_{V,W}$ so that we can replace ${}^{\tau}\Delta$ with Δ^{op} in (1.10). Finite dimensional type **1** U modules are direct sums of their weight spaces (because of our hypothesis on $\Bbbk$) with weights contained in the set $\Lambda = \{q^a\}_{a \in \mathbb{Z}}$. Let $f : \Lambda \times \Lambda \to \Bbbk^{\times}$. We then define for any V and W a linear isomorphism $\widetilde{f} : V \otimes W \to V \otimes W$ such that

$$\widetilde{f}(v \otimes w) = f(\lambda, \mu) v \otimes w \tag{1.17}$$

whenever $v \in V_\lambda$ and $w \in W_\mu$.

Define $\theta^f_{V,W} = \theta_{V,W} \circ \widetilde{f}$. Sadly, not any f will result in a $\theta^f_{V,W}$ so that

$$\Delta(u) \circ \theta^f_{V,W} = \theta^f_{V,W} \circ (\Delta^{op})(u) \tag{1.18}$$

for all $u \in U$. To see what f will work, we first observe that thanks to the equation

$$\Delta(u) \circ \theta_{V,W} = \theta_{V,W} \circ {}^{\tau}\Delta(u) \tag{1.19}$$

the desired equality holds whenever

$$ {}^{\tau}\Delta(u) \circ \widetilde{f} = \widetilde{f} \circ (\Delta^{op})(u). \tag{1.20}$$

To show (1.20) holds for all $u \in U$, it suffices to just check the equality on the generators E, F, and K. The calculation (an exercise, if you get stuck see Jantzen's proof of lemma 3.13 [2]) shows we must have f satisfy

$$f(\lambda, \mu + \alpha_1) = \lambda^{-1} f(\lambda, \mu) \tag{1.21}$$

and

$$f(\lambda + \alpha_1, \mu) = \mu^{-1} f(\lambda, \mu). \tag{1.22}$$

Theorem 1.3. *Let f satisfy* (1.21) *and* (1.22)*. Then*

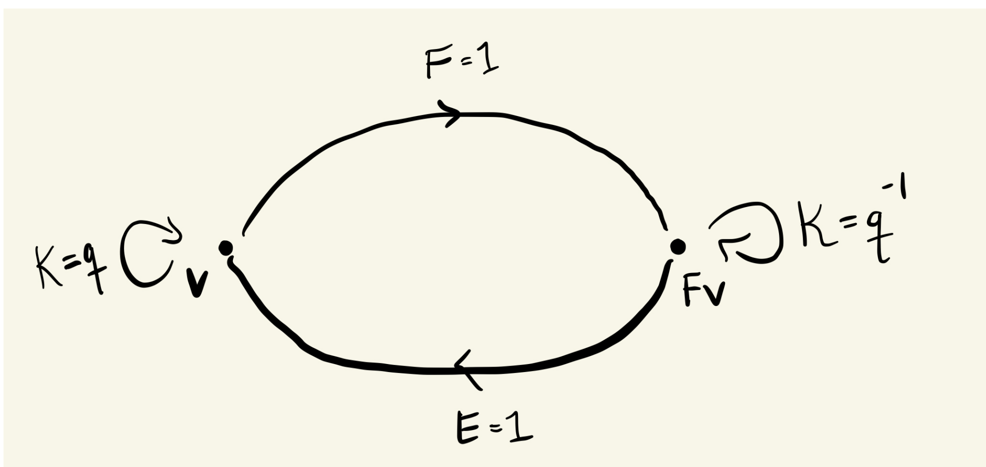
$$R_{V,W} = \theta^f_{V,W} \circ P : V \otimes W \longrightarrow W \otimes V \tag{1.23}$$

is a functorial U-module isomorphism.

Remark 1.4. We saw already that functorality will follow from $R_{V,W}$ being the action of some elements in $U \otimes U$. In order to spell out functorality explicitly, let $\varphi : V \longrightarrow V'$ and $\psi : W \longrightarrow W'$, then

$$(\varphi \otimes \psi) \circ R_{V,W} = R_{V',W'} \circ (\varphi \otimes \psi). \tag{1.24}$$

Example 1.5. The U-module $L(\varpi_1)$ has a basis $\{v, Fv\}$ with action of the generators of U given by



(1.25)

The module $L(\varpi_1) \otimes L(\varpi_1)$ has a basis $\{v \otimes v, v \otimes Fv, Fv \otimes v, Fv \otimes Fv\}$. Note that the elements θ_n act on $L(\varpi_1) \otimes L(\varpi_1)$ as zero for $n \geq 2$, so

$$\theta_{L(\varpi_1),L(\varpi_1)} = 1 - (q - q^{-1}) F \otimes E|_{L(\varpi_1) \otimes L(\varpi_1)}. \tag{1.26}$$

which in our basis is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + (q^{-1} - q) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{1.27}$$

Let $f(\varpi_1, \varpi_1) = q^{-1}$. One can check that this forces $f(-\varpi_1, -\varpi_1) = q^{-1}$ and $f(\varpi_1, -\varpi_1) = 1 = f(-\varpi_1, \varpi)_1$. Computing the matrix of $R_{L(\varpi_1),L(\varpi_1)} = \theta_{L(\varpi_1),L(\varpi_1)} \circ \widetilde{f} \circ P$ we find

$$R_{L(\varpi_1),L(\varpi_1)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & q^{-1}-q & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} q^{-1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.28}$$

so

$$R_{L(\varpi_1),L(\varpi_1)} = \begin{pmatrix} q^{-1} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & q^{-1}-q & 0 \\ 0 & 0 & 0 & q^{-1} \end{pmatrix} \tag{1.29}$$

Remark 1.6. In order for R to be a U-module isomorphism we only needed f to satisfy the conditions (1.21) and (1.22). However, we will want R to satisfy further coherence condition (hexagon equations) which will end up requiring that f satisfy

$$f(\lambda, \mu\nu) = f(\lambda, \mu) f(\lambda, \nu) \quad \text{and} \quad f(\lambda\mu, \nu) = f(\lambda, \nu) f(\lambda, \nu). \tag{1.30}$$

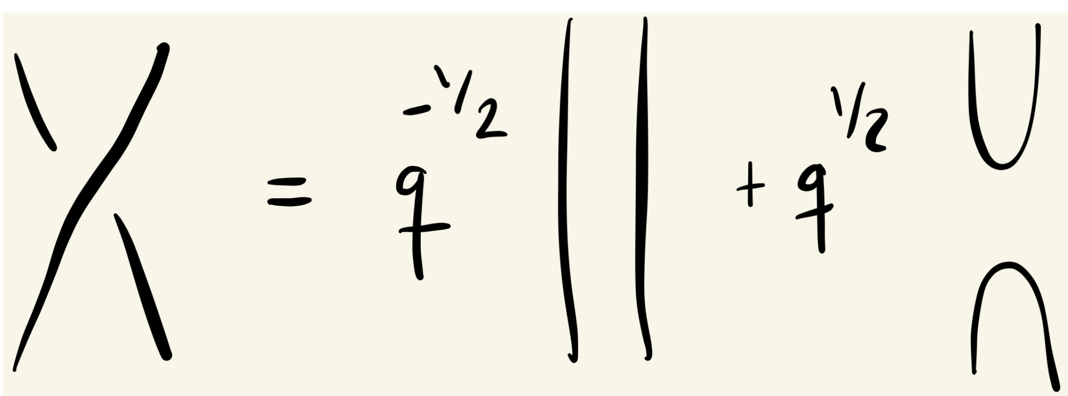
If $\Bbbk$ contains a square root of q, say $v = q^{1/2} \in \Bbbk$ then there is a choice for f which will satisfy all these conditions (for finite dimensional type **1** modules). We set $f(a\varpi, b\varpi) = v^{-ab}$.

$$R_{L(\varpi_1),L(\varpi_1)} = \begin{pmatrix} v^{-1} & 0 & 0 & 0 \\ 0 & 0 & v & 0 \\ 0 & v & v^{-1}-v^3 & 0 \\ 0 & 0 & 0 & v^{-1} \end{pmatrix} \tag{1.31}$$

The representation $L(\varpi_1)$ carries a non-degenerate form which determines two U-module maps

$$\text{cap} : L(\varpi_1) \otimes L(\varpi_1) \to \Bbbk \qquad \text{cup} : \Bbbk \to L(\varpi_1) \otimes L(\varpi_1) \tag{1.32}$$

where $\text{cap}(v \otimes v) = 0 = \text{cap}(Fv \otimes Fv)$, $\text{cap}(v \otimes Fv) = 1$, and $\text{cap}(Fv \otimes v) = -q$, while $\text{cup}(1) = -q^{-1} v \otimes Fv + Fv \otimes v$. These morphisms have a well known diagrammatic description in terms of the Temperley-Lieb category. Giving rise to the skein relation for the Jones polynomial



(1.33)

which exactly agrees with the R matrix above, under the assignment of the cup and cap diagrams with our cup and cap morphisms. In other words, $R_{L(\varpi_1),L(\varpi_1)} = v^{-1}\,\text{id} + v\text{cup} \circ \text{cap}$.

1.3. **The $\mathfrak{g}$ case.** Let Φ be a root system with choice of simple roots Π and symmetric form $(-,-)$ so that $(\alpha,\alpha)=2$ for all short roots (long roots pair with themselves to be 4 or 6, the latter only for G_2 components). Then we obtain $U=U_q(\mathfrak{g})$ which is a $\Bbbk$-algebra defined by generators and relations.

By placing the operators E_α in degree α and F_α in degree $-\alpha$, the algebra U is graded by $\mathbb{Z}\Phi$. Furthermore, the subalgebras U^+ and U^- (generated by the $E's$ and $F's$ respectively) are graded subalgebras.

In the previous lecture we learned:

Proposition 1.7. *There is a unique bilinear pairing $(-,-) : U^{\leq 0}\times U^{\geq 0}\to \Bbbk$ such that for all $x,x'\in U^{\geq 0}$, all $y,y'\in U^{\leq 0}$, all $\mu,\nu\in\mathbb{Z}\Phi$, and all $\alpha,\beta\in\Pi$*

$$(y,xx')=(\Delta(y),x'\otimes x),\qquad (yy',x)=(y\otimes y',\Delta(x)),\tag{1.34}$$

$$(K_\mu,K_\nu)=q^{-(\mu,\nu)},\qquad (F_\alpha,F_\beta)=-\delta_{\alpha\beta}(q_\alpha-q_\alpha^{-1})^{-1},\tag{1.35}$$

$$(K_\mu,E_\alpha)=0,\qquad (F_\alpha,K_\mu)=0.\tag{1.36}$$

Proposition 1.8. *Assume that $\Bbbk$ is characteristic zero and q is transcendental over $\mathbb{Q}$. The restriction of $(-,-)$ to any $U^-_{-\mu}\times U^+_\mu$ with $\mu\in\mathbb{Z}\varphi$, $\mu\geq 0$ is a nondegenerate pairing.*

For each $\mu\in\mathbb{Z}_{\geq 0}\Phi_+$ choose a basis $u_1^\mu,\dots,u^\mu_{r(\mu)}$ of U^+_μ. Then there is a dual basis $v_1^\mu,\dots,v^\mu_{r(\mu)}$ of $U^-_{-\mu}$ so that $(v_j^\mu,u_i^\mu)=\delta_{ij}$. Set

$$\theta_\mu=\sum_{i=1}^{r(\mu)} v_i^\mu\otimes u_i^\mu\in U\otimes U.\tag{1.37}$$

Example 1.9. The bilinear form is given on $U_q(\mathfrak{sl}_2)$ by

$$(F^n,E^n)=\frac{(-1)^n q^{n(n-1)/2}[n]!}{(q-q^{-1})^n}\tag{1.38}$$

while $U_{n\alpha}=\Bbbk\cdot E^n$ and $U_{-n\alpha}=\Bbbk\cdot F^n$. Thus we can take our dual bases to be

$$E^n\qquad\text{and}\qquad \frac{(-1)^n q^{-n(n-1)/2}(q-q^{-1})^n}{[n]!}F^n\tag{1.39}$$

so

$$\theta_{n\alpha}=\frac{(-1)^n q^{-n(n-1)/2}(q-q^{-1})^n}{[n]!}F^n\otimes E^n\tag{1.40}$$

Exercise 1.10. Compute $\theta_{\alpha+\beta}$ for $U=U_q(\mathfrak{sl}_3)$. Hint: Use that $U^+\cong\Bbbk\langle E_\alpha,E_\beta\mid \text{q-Serre relation}\rangle$, so $U^+_{\alpha+\beta}$ has basis $\{E_\alpha E_\beta,E_\beta E_\alpha\}$. Similarly, $U^-_{-(\alpha+\beta)}$ has basis $\{F_\alpha F_\beta,F_\beta F_\alpha\}$. Then use the definition of the pairing $(-,-)$ to compute its restriction to $U^-_{-(\alpha+\beta)}\times U^+_{\alpha+\beta}$.

Remark 1.11. The element θ_μ does not depend on our choice of basis u_i^μ.

Define, as we did for $U_q(\mathfrak{sl}_2)$, ${}^\tau\Delta=\tau\otimes\tau\circ\Delta\circ\tau^{-1}$.

Lemma 1.12.

$$\Delta(u)\circ\theta_{V,W}=\theta_{V,W}\circ{}^\tau\Delta(u)\tag{1.41}$$

for all $u\in U$.

Proof. We need to argue that

$$(E_\alpha \otimes 1)\theta_\mu + (K_\alpha \otimes E_\alpha)\theta_{\mu-\alpha} = \theta_\mu(E_\alpha \otimes 1) + \theta_{\mu-\alpha}(K_\alpha^{-1} \otimes E_\alpha), \tag{1.42}$$

$$(1 \otimes F_\alpha)\theta_\mu + (F_\alpha \otimes K_\alpha^{-1})\theta_{\mu-\alpha} = \theta_\mu(1 \otimes F_\alpha) + \theta_{\mu-\alpha}(F_\alpha \otimes K_\alpha), \tag{1.43}$$

$$(K_\alpha \otimes K_\alpha)\theta_\mu = \theta_\mu(K_\alpha \otimes K_\alpha) \tag{1.44}$$

To see (1.44) observe that the $\theta_\mu' s$ have degree zero in the $\mathbb{Z}\Phi$ grading.

We will prove (1.42) as the proof of (1.43) is similar. First, we note that the usual dual basis technology tells us we can write for $u \in U_\mu^+$

$$u = \sum (v_i^\mu, u)u_i^\mu \tag{1.45}$$

and for $v \in U_{-\mu}^-$

$$v = \sum (v, u_i^\mu)v_i^\mu. \tag{1.46}$$

Then we recall that $q_\alpha = q^{(\alpha,\alpha)/2}$, and set $c_\alpha = \dfrac{1}{q_\alpha - q_\alpha^{-1}}$.

We need to briefly review some useful functions defined when we were studying the form $(-,-)$. For $x \in U_\mu^+$ (still with $\mu \geq 0$) we have

$$\Delta(x) \in \bigoplus_{0 \leq \nu \leq \mu} U_{\mu-\nu}^+ K_\nu \otimes U_\nu^+ \tag{1.47}$$

so

$$\Delta(x) = x \otimes 1 + \sum_{\alpha \in \Delta} r_\alpha(x) K_\alpha \otimes E_\alpha + \text{(rest)} \tag{1.48}$$

defines for us $r_\alpha(x) \in U_{\mu-\alpha}^+$, and

$$\Delta(x) = K_\mu \otimes x + \sum_{\alpha \in \Delta} E_\alpha K_{\mu-\alpha} \otimes r'_\alpha(x) + \text{(rest)} \tag{1.49}$$

defines $r'_\alpha(x) \in U_{\mu-\alpha}^+$.

We can also define, for $y \in U_{-\mu}^-$, elements $r_\alpha(y) \in U_{-(\mu-\alpha)}^-$ and $r'_\alpha(y) \in U_{-(\mu-\alpha)}^-$ by observing

$$\Delta(y) \in \bigoplus_{0 \leq \nu \leq \mu} U_{-\nu}^- \otimes U_{-(\mu-\nu)}^- K_\nu^{-1} \tag{1.50}$$

so

$$\Delta(y) = y \otimes K_\mu^{-1} + \sum_{\alpha \in \Delta} r_\alpha(y) \otimes F_\alpha K_{\mu-\alpha}^{-1} + \text{(rest)} \tag{1.51}$$

and

$$\Delta(y) = 1 \otimes y + \sum_{\alpha \in \Delta} F_\alpha \otimes r'_\alpha(y) K_\alpha^{-1} + \text{(rest)}. \tag{1.52}$$

There are many identities among the $r_\alpha(x)$ but we only need the following two to prove (1.42). They are [2] $6.17(1)$ and $6.15(5)$: for $x \in U_\mu^+$ and $y \in U_\mu^-$

$$E_\alpha y - y E_\alpha = c_\alpha \left(K_\alpha r_\alpha(y) - r'_\alpha(y) K_\alpha^{-1} \right), \tag{1.53}$$

$$(y, E_\alpha x) = (F_\alpha, E_\alpha)(r_\alpha(y), x) = -c_\alpha (r_\alpha(y), x), \tag{1.54}$$

and

$$(y, xE_\alpha) = (F_\alpha, E_\alpha)(r'_\alpha(y), x) = -c_\alpha(r'_\alpha(y), x). \tag{1.55}$$

Finally, we compute that

$$\begin{aligned}
(E_\alpha \otimes 1)\theta_\mu - \theta_\mu(E_\alpha \otimes 1) &= \sum_i (E_\alpha v_i^\mu - v_i^\mu E_\alpha) \otimes u_i^\mu \\
&= c_\alpha \sum_i \left(K_\alpha r_\alpha(v_i^\mu) - r'_\alpha(v_i^\mu) K_\alpha^{-1}\right) \otimes u_i^\mu \\
&= c_\alpha \sum_i \left(K_\alpha \sum_j (r_\alpha(v_i^\mu), u_j^{\mu-\alpha}) v_j^{\mu-\alpha} - \sum_j (r'_\alpha(v_i^\mu), u_j^{\mu-\alpha}) v_j^{\mu-\alpha} K_\alpha^{-1} \right) \otimes u_i^\mu \\
&= \sum_i \left(-K_\alpha \sum_j (v_i^\mu, E_\alpha u_j^{\mu-\alpha}) v_j^{\mu-\alpha} + \sum_j (v_i^\mu, u_j^{\mu-\alpha} E_\alpha) v_j^{\mu-\alpha} K_\alpha^{-1} \right) \otimes u_i^\mu \\
&= \sum_i \left(+ \sum_j (v_i^\mu, u_j^{\mu-\alpha} E_\alpha) v_j^{\mu-\alpha} K_\alpha^{-1} - K_\alpha \sum_j (v_i^\mu, E_\alpha u_j^{\mu-\alpha}) v_j^{\mu-\alpha} \right) \otimes u_i^\mu \\
&= \left(\sum_j (v_i^\mu, u_j^{\mu-\alpha} E_\alpha) v_j^{\mu-\alpha} K_\alpha^{-1} \otimes \sum_i u_i^\mu \right) - \left(K_\alpha \sum_j (v_i^\mu, E_\alpha u_j^{\mu-\alpha}) v_j^{\mu-\alpha} \otimes \sum_i u_i^\mu \right) \\
&= \left(\sum_j v_j^{\mu-\alpha} K_\alpha^{-1} \otimes \sum_i (v_i^\mu, u_j^{\mu-\alpha} E_\alpha) u_i^\mu \right) - \left(K_\alpha \sum_j v_j^{\mu-\alpha} \otimes \sum_i (v_i^\mu, E_\alpha u_j^{\mu-\alpha}) u_i^\mu \right) \\
&= \left(\sum_j v_j^{\mu-\alpha} K_\alpha^{-1} \otimes u_j^{\mu-\alpha} E_\alpha \right) - \left(K_\alpha \sum_j v_j^{\mu-\alpha} \otimes E_\alpha u_j^{\mu-\alpha} \right) \\
&= \sum_j v_j^{\mu-\alpha} K_\alpha^{-1} \otimes u_j^{\mu-\alpha} E_\alpha - K_\alpha v_j^{\mu-\alpha} \otimes E_\alpha u_j^{\mu-\alpha} \\
&= \left(\sum_j v_j^{\mu-\alpha} \otimes u_j^{\mu-\alpha} \right) K_\alpha^{-1} \otimes E_\alpha - K_\alpha \otimes E_\alpha \left(\sum_j v_j^{\mu-\alpha} \otimes u_j^{\mu-\alpha} \right) \\
&= \theta_{\mu-\alpha}(K_\alpha^{-1} \otimes E_\alpha) - (K_\alpha \otimes E_\alpha)\theta_{\mu-\alpha}.
\end{aligned}$$

□

If V and W are finite dimensional (type **1**) U-modules, then both are direct sums of their weight spaces and

$$\theta_\mu : V_\lambda \otimes W_{\lambda'} \longrightarrow V_{\lambda-\mu} \otimes W_{\lambda'+\mu}. \tag{1.56}$$

Since W and V are finite dimensional, we can define the action of

$$\theta_{V,W} = \sum_{\mu \geq 0} \theta_\mu|_{V \otimes W} \tag{1.57}$$

Again, we can choose ordered bases so that $\sum_{\mu>0} \theta_\mu$ acts as a strictly upper triangular operator on $V \otimes W$. Since $\theta_0 = 1 \otimes 1$, θ is a unipotent endomorphism of $V \otimes W$.

We will again define $\theta^f = \theta \circ \widetilde{f}$ where $\widetilde{f}$ is derived from some function $f : \Lambda \times \Lambda \to \Bbbk^\times$ satisfying some property analogous to the $U_q(\mathfrak{sl}_2)$ case.

Theorem 1.13. *Suppose that $f : \Lambda \times \Lambda \to \Bbbk^\times$ is such that for all $\lambda, \mu \in \Lambda$ and $\nu \in \mathbb{Z}\Phi$*

$$f(\lambda + \nu, \mu) = q^{-(\nu,\mu)}(f(\lambda, \mu) \tag{1.58}$$

and

$$f(\lambda, \mu + \nu) = q^{-(\nu,\lambda)} f(\lambda, \mu). \tag{1.59}$$

Then the map

$$R_{V,W} = \theta^f \circ P : V \otimes W \to W \otimes V \tag{1.60}$$

is an isomorphism of U-modules. Furthermore, $\theta_{V,W}$ is functorial in V and W.

Proof. Similar to the $\mathfrak{sl}_2$ case. □

Remark 1.14. In order for R to be a U-module isomorphism we need f to satisfy (1.58) and (1.59). But, in order for R to solve the hexagon equation we will need further conditions on f.

Let $d = [\Lambda : \mathbb{Z}\Phi]$. Suppose that $\Bbbk$ contains a d-th root of unity of q, denoted v, and set $f(\lambda, \nu) = v^{-d \cdot (\lambda,\nu)}$. Then this f will give rise to an R which is an honest braiding on the category of finite dimensional type **1** U-modules.

Example 1.15. If the simple roots for $\mathfrak{sl}_3$ are α, β then the fundamental weights are $\varpi_1 = \frac{1}{3}(2\alpha + \beta)$ and $\varpi_2 = \frac{1}{3}(\alpha + 2\beta)$. Using $(-,-)$ for the W-invariant bilinear form on $\mathbb{Z}\Phi$ so that $(\alpha, \alpha) = 2 = (\beta, \beta)$, we find

$$(\varpi_1, \varpi_1) = \frac{2}{3} = (\varpi_2, \varpi_2) \tag{1.61}$$

and

$$(\varpi_1, \varpi_2) = \frac{1}{3} = (\varpi_2, \varpi_1). \tag{1.62}$$

Therefore,

$$f((a, b), (c, d)) = q^{-\frac{1}{3} \cdot (2ac+bc+ab+2bd)}. \tag{1.63}$$

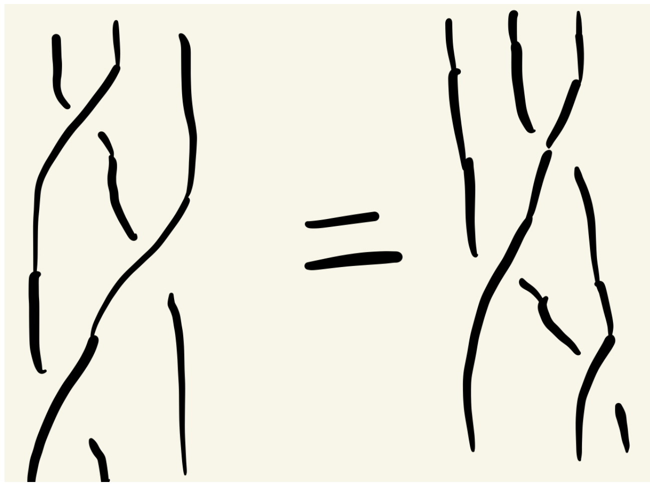
For $U_q(\mathfrak{sl}_n)$ we will need $q^{\frac{1}{n}} \in \Bbbk$.

2. Hexagon Equations and the Braid Group

2.1. **Motivating Coherence for the R matrix.** Recall that the braid group on n strands has the following generators and relations presentation:

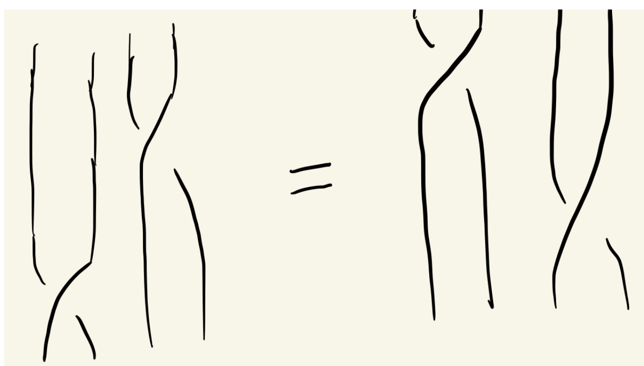
$$Br_n = \langle \sigma_1, ..., \sigma_{n-1} \mid \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \sigma_i \sigma_j = \sigma_j \sigma_i \text{ when } |i - j| > 1 \rangle, \tag{2.1}$$

where σ_i maps to the (isotopy class of) the ″positive crossing of the i-th and $i+1$-st strands. The relation $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ can be visualized locally in Br_n as:



(2.2)

The relation $\sigma_i\sigma_j = \sigma_j\sigma_i$ can be visualized in Br_n as



(2.3)

Given an endomorphism R in $\mathrm{End}_U(V \otimes V)$ we can define U-module endomorphisms $R_{12} = R \otimes \mathrm{id}$ and $R_{23} = \mathrm{id} \otimes R$ of $V \otimes V \otimes V$. If R satisfies

$$R_{12} \circ R_{23} \circ R_{12} = R_{23} \circ R_{12} \circ R_{23}. \tag{2.4}$$

We will see that setting $R = \theta^f_{V,V} \circ P$ will give a solution to (2.4). (One reason to care is that this gives rise to some linear representations of type A braid group. These were quite scarce before quantum groups.)

So far we have shown that the $R_{V,W}$ are *functorial* U-module isomorphisms $V \otimes W \xrightarrow{\sim} W \otimes V$. We have two types of composition of morphisms: the usual function composition $\varphi \circ \psi$ as well as the tensor product of morphisms $\varphi \otimes \psi$. So we might want two types of consistency for the braiding. The first is functorality and the second type of consistency we want is the *hexagon equations*

(2.5)

$$\begin{tikzcd}
 & M_1 \otimes (M_3 \otimes M_2) \arrow[r, "\mathrm{can}"] & (M_1 \otimes M_3) \otimes M_2 \arrow[dr, "R"] & \\
M_1 \otimes (M_2 \otimes M_3) \arrow[ur, "R"] \arrow[dr, "\mathrm{can}"'] & & & (M_3 \otimes M_1) \otimes M_2 \\
 & (M_1 \otimes M_2) \otimes M_3 \arrow[r, "R"'] & M_3 \otimes (M_1 \otimes M_2) \arrow[ur, "\mathrm{can}"'] &
\end{tikzcd}$$

(2.6)

$$\begin{tikzcd}
 & (M_2 \otimes M_1) \otimes M_3 \arrow[r, "\mathrm{can}"] & M_2 \otimes (M_1 \otimes M_3) \arrow[dr, "R"] & \\
(M_1 \otimes M_2) \otimes M_3 \arrow[ur, "R"] \arrow[dr, "\mathrm{can}"'] & & & M_2 \otimes (M_3 \otimes M_1) \\
 & M_1 \otimes (M_2 \otimes M_3) \arrow[r, "R"'] & (M_2 \otimes M_3) \otimes M_1 \arrow[ur, "\mathrm{can}"'] &
\end{tikzcd}$$

Remark 2.1. Note that can is the usual vector space isomorphism between triple tensor products which re-associates simple tensors. For example $\mathrm{can}(m \otimes (m' \otimes m'')) = (m \otimes m') \otimes m''$. These maps all commute with the action of U, since the comultiplication $\Delta : U \longrightarrow U \otimes U$ is coassociative. In the proofs below we will ignore these maps.

Remark 2.2. Once we show that $R_{V,W}$ is a family of funtorial isomorphisms satisfying the Hexagon equations, we will have shown that $U - \mathrm{mod}^{\mathrm{type}\,\mathbf{1}}$ is a braided tensor category. Intuitively, this just means that any two morphism built out of the canonical morphisms and the $R_{V,W}$'s are equal if they represent the same element of the braid group. For more precise discussion of braided tensor categories see [1]

Proposition 2.3. *For all $V \in U - \mathit{mod}^{\mathit{type}\,\mathbf{1}}$, we have the following equality in $\mathit{End}_U(V \otimes V \otimes V)$*

(2.7) $$(\mathrm{id} \otimes R_{V,V}) \circ (R_{V,V} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes R_{V,V}) = (R_{V,V} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes R_{V,V}) \circ (R_{V,V} \otimes \mathrm{id}).$$

Proof. Using the Hexagon equation and functorality we find

$$\begin{aligned}
(\mathrm{id} \otimes R_{V,V}) \circ (R_{V,V} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes R_{V,V}) &= (\mathrm{id} \otimes R_{V,V}) \circ R_{V \otimes V, V} \\
&= R_{V \otimes V, V} \circ (R_{V,V} \otimes \mathrm{id}) \\
&= (R_{V,V} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes R_{V,V}) \circ (R_{V,V} \otimes \mathrm{id}).
\end{aligned}$$

□

Remark 2.4. Note that (2.4) is different than the equation $\theta^f_{23}\theta^f_{13}\theta^f_{12} = \theta^f_{12}\theta^f_{23}\theta^f_{23}$, which is what is more properly called the quantum Yang-Baxter equation. As long as f satisfies the conditions so that $\theta^f P$ is a U module isomorphism, we will have that θ^f solves the quantum

Yang Baxter equation. Then one shows that $\theta^f P$ satisfies braid relation (2.4) using that θ^f satisfies the quantum Yang-Baxter equation.

However, if f is so that $\theta^f P$ satisfies the Hexagon equation, then $\theta^f P$ will satisfy the braid relation (2.4). Jantzen handles the more general situation in [2], but for ease of exposition we just assume f is so that $\theta^f P$ satisfies the Hexagon equation from the start, then use this to deduce (2.4).

In both cases the R matrix gives rise to a braid group action, but only in the latter case does the R matrix give rise to a braiding on $U - \text{mod}^{\text{type}\mathbf{1}}$.

2.2. **Proving the Hexagon Identity.** Based on the exercise, if our goal is for R to satisfy the braid relation (2.4), we must derive the hexagon equations for R.

Theorem 2.5. *Let M_1, M_2, and M_3 be finite dimensional, type $\mathbf{1}$, $U_q(\mathfrak{g})$-modules. Let f satisfy* (1.21), (1.22), *and* (1.30). *Then both diagrams in the hexagon equations commute.*

Proof. We will show the first diagram commutes, the argument for the second diagram being similar.

The top half of the diagram is

$$(\theta \otimes 1) \circ (\widetilde{f} \otimes 1) \circ (P \otimes 1) \circ (1 \otimes \theta) \circ (1 \otimes \widetilde{f}) \circ (1 \otimes P). \tag{2.8}$$

If $\theta = \sum a_i \otimes b_i$, then we define $\theta_{13} = \sum a_i \otimes 1 \otimes b_i$, and then can write

$$\theta_{13} = (P \otimes 1) \circ (1 \otimes \theta) \circ (P \otimes 1). \tag{2.9}$$

If we define $\widetilde{f}_{13}$ so that for $m \otimes m' \otimes m'' \in M_{\lambda_1} \otimes M'_{\lambda_2} \otimes M''_{\lambda_3}$, $\widetilde{f}_{13}(m \otimes m' \otimes m'') = f(\lambda_1, \lambda_3) m \otimes m' \otimes m''$, then we can write

$$\widetilde{f}_{13} = (P \otimes 1) \circ (1 \otimes \widetilde{f}) \circ (P \otimes 1). \tag{2.10}$$

We define

$$\theta' = \sum_{\mu} (1 \otimes K_\mu \otimes 1) \circ (\theta_\mu)_{13}. \tag{2.11}$$

Using that $\theta_\mu : V_{\lambda_1} \otimes W_{\lambda_2} \longrightarrow V_{\lambda_1 - \mu} \otimes W_{\lambda_2 + \mu}$ and $\widetilde{f}(\lambda_1 - \mu, \lambda_2) = q^{(\mu, \lambda_2)} \widetilde{f}(\lambda_1, \lambda_2)$, it follows that

$$(1 \otimes K_\mu \otimes 1) \circ (\theta_\mu)_{13} \circ (\widetilde{f} \otimes 1) = (\widetilde{f} \otimes 1) \circ (\theta_\mu)_{13}. \tag{2.12}$$

Thus,

$$\theta' \circ (\widetilde{f} \otimes 1) = (\widetilde{f} \otimes 1) \circ \theta_{13}. \tag{2.13}$$

Using (2.9), (2.10), and (2.13) we may rewrite the top half of the diagram (2.8) as

$$(\theta \otimes 1) \circ \theta' \circ (\widetilde{f} \otimes 1) \circ \widetilde{f}_{13} \circ (P \otimes 1) \circ (1 \otimes P). \tag{2.14}$$

The bottom half of the diagram is

$$R_{M \otimes M', M''} = \theta_{M'', M \otimes M'} \circ \widetilde{f} \circ P_{M \otimes M', M''} = (1 \otimes \Delta)(\theta) \circ \widetilde{f} \circ (P \otimes 1) \circ (1 \otimes P). \tag{2.15}$$

By $\widetilde{f}$ we mean for $m'' \otimes (m \otimes m') \in M''_{\lambda_3} \otimes (M \otimes M')_{\lambda_1 + \lambda_2}$

$$\widetilde{f}(m'', m \otimes m') = f(\lambda_3, \lambda_1 + \lambda_2) m'' \otimes m \otimes m'. \tag{2.16}$$

But since $f(\lambda_3, \lambda_1 + \lambda_2) = f(\lambda_3, \lambda_1) f(\lambda_3, \lambda_2)$ we have

$$\widetilde{f}(m'', m \otimes m') = (\widetilde{f} \otimes 1) \circ \widetilde{f}_{13}(m'' \otimes m \otimes m'). \tag{2.17}$$

We claim that

$$(1 \otimes \Delta)(\theta) = (\theta \otimes 1) \circ \theta'. \tag{2.18}$$

Once, we establish (2.18), it is easy to see that we can rewrite (2.15) as

$$(1 \otimes \Delta)(\theta) \circ \widetilde{f} \circ (P \otimes 1) \circ (1 \otimes P) = (\theta \otimes 1) \circ \theta' \circ (\widetilde{f} \otimes 1) \circ \widetilde{f}_{13} \circ (P \otimes 1) \circ (1 \otimes P) \tag{2.19}$$

proving that the top half and bottom half of the diagram are equal.

We will deduce (2.18) once we show that

$$(1 \otimes \Delta)(\theta_\mu) = \sum_{0 \leq \nu \leq \mu} (\theta_{\mu-\nu} \otimes 1) \circ (1 \otimes K_\nu \otimes 1) \circ (\theta_\nu)_{13}. \tag{2.20}$$

To this end, recall that for $x \in U^+$,

$$\Delta(x) \in \bigoplus_{0 \leq \nu \leq \mu} U^+_{\mu-\nu} K_\nu \otimes U^+_\nu \tag{2.21}$$

from which it follows that

$$\Delta(x) = \sum_{\nu,i,j} c^\nu_{ij} u_i^{\mu-\nu} K_\nu \otimes u^\nu_j. \tag{2.22}$$

The scalars c^ν_{ij} can be computed using the dual basis as

$$c^\nu_{ij} = (v_i^{\mu-\nu} \otimes v_j, \Delta(x)) = v(_i^{\mu-\nu} v^\nu_j, x), \tag{2.23}$$

which implies the following formula

$$\Delta(x) = \sum_{0 \leq \nu \leq \mu} \sum_{i,j} (v_i^{\mu-\nu} v^\nu_j, x) u_i^{\mu-\nu} K_\nu \otimes u^\nu_j. \tag{2.24}$$

Finally, we recall that by definition

$$\theta_\mu = \sum_i v^\mu_i \otimes u^\mu_i, \tag{2.25}$$

so we can use (2.24) to write

$$\begin{aligned}
(1 \otimes \Delta)(\theta_\mu) &= \sum_i v^\mu_i \otimes \Delta(u^\mu_i) \\
&= \sum_i v^\mu_i \otimes \sum_{\nu,p,q} (v_p^{\mu-\nu} v^\nu_q, u^\mu_i) u_p^{\mu-\nu} K_\nu \otimes u^\nu_q \\
&= \sum_{\nu,p,q} \left(\sum_i (v_p^{\mu-\nu} v^\nu_q, u^\mu_i) v^\mu_i \right) \otimes u_p^{\mu-\nu} K_\nu \otimes u^\nu_q \\
&= \sum_{\nu,p,q} v_p^{\mu-\nu} v^\nu_q \otimes u_p^{\mu-\nu} K_\nu \otimes u^\nu_q \\
&= \sum_{0 \leq \nu \leq \mu} (\theta_{\mu-\nu} \otimes 1) \circ (1 \otimes K_\nu \otimes 1) \circ (\theta_\nu)_{13}.
\end{aligned}$$

□

2.3. **Hecke Algebras and Quantum Schur Weyl Duality.** The Hecke algebra $\mathbb{H}_d$ is the quotient of $\Bbbk Br_d$ by the ideal generated by

$$\sigma_i^2 = (q^{-1} - q)\sigma_i + 1. \tag{2.26}$$

We will denote the image of the generator σ_i in the quotient by H_i.

Let V_1 denote the vector representation of the Lie algebra $\mathfrak{sl}_n$ and let $V_q = L(\varpi_1)$ denote the quantized vector representation, or first fundamental representation, for $U_q(\mathfrak{sl}_n)$.

Note that the symmetric group is a quotient of the braid group by a simpler quadratic relation $\sigma_i^2 = 1$. The usual Schur-Weyl duality says that the action of S_d on $V_1^{\otimes d}$ which permutes the tensors, will induce a surjective algebra homomorphism $\mathbb{C}S_d \to \mathrm{End}_{\mathfrak{sl}_n}(V_1^{\otimes d})$.

The action of the symmetric group on $V_1^{\otimes d}$ is generated by the endomorphisms $s_i = \mathrm{id}_{i-1} \otimes P \otimes \mathrm{id}_{i+1}$. So it is natural to consider what algebra is generated by the action of $R_i = \mathrm{id}_{i-1} \otimes R \otimes \mathrm{id}_{i+1}$ on the $U_q(\mathfrak{sl}_n)$-module $V_q^{\otimes d}$. We know that the R_i satisfy the hexagon equation, and therefore the braid relation. Also, the usual interchange law for morphisms between tensor products tells us that the maps $\mathrm{id} \otimes R \otimes \mathrm{id}$ satisfy the second relation in the braid group (distant braids commute). Therefore, whatever algebra the action generates will be a quotient of the braid group.

Example 2.6. Recall that for $\mathfrak{sl}_2$ we found that after writing $v = q^{1/2}$

$$R_{V_q,V_q} = \begin{pmatrix} v^{-1} & 0 & 0 & 0 \\ 0 & 0 & v & 0 \\ 0 & v & v^{-1} - v^3 & 0 \\ 0 & 0 & 0 & v^{-1} \end{pmatrix} \tag{2.27}$$

We compute $(R_{V_q,V_q} - v^{-1})(R_{V_q,V_q} + v^3)$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -v^{-1} & v & 0 \\ 0 & v & -v^3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} v^{-1} + v^3 & 0 & 0 & 0 \\ 0 & v^3 & v & 0 \\ 0 & v & v^{-1} & 0 \\ 0 & 0 & 0 & v^{-1} + v^3 \end{pmatrix}. \tag{2.28}$$

which is easily seen to be zero. Therefore, $R^2 = v^2 \,\mathrm{id} + (v^{-1} - v^3)R$. and it follows that setting $H = v^{-1}R$ we get

$$H^2 = v^{-2}R^2 = (v^{-2} - v^2)H + \mathrm{id} = (q^{-1} - q)H + \mathrm{id}\,. \tag{2.29}$$

This proves that for $U_q(\mathfrak{sl}_2)$, the action of the R_i's on $V_q^{\otimes d}$ gives an action of Br_d which factors through the algebra $\mathbb{H}_d$. Furthermore, one can argue that the induced homomorphism

$$\mathbb{H}_d \longrightarrow \mathrm{End}_{U_q(\mathfrak{sl}_2)}(V_q^{\otimes d}) \tag{2.30}$$

is surjective.

The calculation of the quadratic relation generalizes to $U_q(\mathfrak{sl}_n)$ and V_q. Again, one finds that in order to define R_{V_q,V_q} the field $\Bbbk$ must contain an element v so that $v^n = q$. But, the quadratic relation will then be $H^2 = (v^{-n} - v^n)H + 1 = (q^{-1} - q)H + 1$.

Thus, for $V_q^{\otimes d}$ it is still the algebra $\mathbb{H}_d$ acting. This action $\mathbb{H}_d \to \mathrm{End}_{U_q(\mathfrak{sl}_n}(V_q^{\otimes d})$ always generates the Endomorphism ring. When $n \geq d$, the action is faithful, and when $n < d$ the kernel is understood.

When q is transcendental the result will follow from the calssical Schur Weyl duality and a deformation argument. However, the result is still true when q is a root of unity.

3. THE QUANTIZED COORDINATE RING $\Bbbk_q[G]$

3.1. **Motivation: The classical coordinate ring $\mathbb{C}[G]$.** More detail can be found in chapter seven of [3]. For simplicity we work over the complex numbers in this motivational interlude. Given a finite dimensional semisimple Lie algebra $\mathfrak{g}$, there is an associated connected and simply connected algebraic group G. The group G is an affine algebraic variety and we write $\mathbb{C}[G]$ to denote the *coordinate ring* of G.

Define the *derivations* of G to be

$$\mathrm{Der}(G) = \{X \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}[G]) \mid X(fg) = X(f)g + fX(g)\}. \tag{3.1}$$

These are vector fields on G, or first order differential operators. We define the *left invariant derivations* to be

$$L(G) = \{X \in \mathrm{Der}(G) \mid X \circ \ell_g = \ell_g \circ X \text{ for all } g \in G\}. \tag{3.2}$$

where $\ell_g : \mathbb{C}[G] \longrightarrow \mathbb{C}[G]$ maps $f \mapsto (x \mapsto f(g^{-1}x))$ and define the *ring of left invariant differential operators*, denoted $D_\ell(G)$, to be the subalgebra of $\mathrm{End}_{\mathbb{C}}(\mathbb{C}[G])$ generated by $L(G)$. The left invariant vector fields on G are identified with the tangent space of G at the identity i.e. the Lie algebra $\mathfrak{g}$. Thus, there is an algebra homomorphism

$$U(\mathfrak{g}) \longrightarrow D_\ell(G). \tag{3.3}$$

A theorem of Cartier says that since we are working over a field of characteristic zero, this is an algebra isomorphism. Thus, we have a pairing

$$U(\mathfrak{g}) \times \mathbb{C}[G] \longrightarrow \mathbb{C} \tag{3.4}$$

defined as $\langle D, f\rangle = D(f)(1)$, *differentiate and evaluate at the identity*. It is an exercise to show this pairing is non-degenerate. In particular, there is an embedding

$$\mathbb{C}[G] \longrightarrow U(\mathfrak{g})^*, \ f \mapsto \langle -, f\rangle \tag{3.5}$$

(For the exercise, to prove the kernel of $D \mapsto \langle D, -\rangle$ is injective Jantzen suggests using Krull intersection theorem. Note that G is connected.)

Suppose that A is a Hopf algebra. Then we can define a convolution product on A^*. For f, g two linear forms on A we define their product fg as the composition

$$A \xrightarrow{\Delta} A \otimes A \xrightarrow{f\otimes g} \Bbbk \otimes \Bbbk \xrightarrow{\alpha\otimes\beta\mapsto\alpha\beta} \Bbbk. \tag{3.6}$$

Explicitly, for $a \in A$ we have

$$\text{if } \Delta(a) = \sum b_i \otimes c_i, \text{ then } fg(a) = \sum f(b_i)g(c_i). \tag{3.7}$$

The coassociative condition on Δ implies this multiplication is associative. The counit axiom for ϵ implies the functional $\epsilon : a \mapsto 1$ is a unit for the product on A^*.

Definition 3.1. Let V be a finite dimensional $U(\mathfrak{g})$ module. For $v \in V$ and $f \in V^*$ we define the *matrix coefficient* $c_{f,v} \in U(\mathfrak{g})^*$ by

$$c_{f,v}(u) = f(uv) \tag{3.8}$$

for all $u \in U(\mathfrak{g})$.

Lemma 3.2. *Let V and W be finite dimensional $U(\mathfrak{g})$-modules. Then in $U(\mathfrak{g})^*$ we have*

$$c_{f,v}c_{g,w} = c_{f\otimes g, v\otimes w} \tag{3.9}$$

for all $v \in V$, $f \in V^*$, $w \in W$, and $g \in W^*$. and

$$c_{f,v} + c_{g,w} = c_{f\oplus g,(v,w)} \tag{3.10}$$

Proof. Exercise. □

The main point is that the image of $\Bbbk[G]$ in $U(\mathfrak{g})^*$ is the subspace spanned by matrix coefficients of $\mathfrak{g}$ modules. To see this, observe that $\Bbbk[G]$ is spanned by matrix coefficients of G-modules (this follows from the algebraic Peter-Weyl theorem and uses that since G is semisimple G is reductive). Then chasing through definitions we see that the claim follows if every finite dimensional $\mathfrak{g}$ module lifts to a G module, which is why we needed to assume that G was simply connected.

3.2. **Definition of $\Bbbk_q[G]$.** If we did not have a group G to begin with bu did know $U(\mathfrak{g})$, we could have discovered $\mathbb{C}[G]$ as the subalgebra of $U(\mathfrak{g})$ spanned by matrix coefficients of finite dimensional $\mathfrak{g}$ modules. In the quantum case, there is no group but we do have the algebra $U_q(\mathfrak{g})$.

Definition 3.3. Let $U = U_q(\mathfrak{g})$ and let G be the connected, simply connected, semisimple group with Lie algebra $\mathfrak{g}$. The *quantized coordinate algebra of* G, denoted $\Bbbk_q[G]$, is the subalgebra of U^* spanned by matrix coefficients of finite dimensional representations of U.

The lemma (3.2) implies that the subspace of U^* spanned by all $c_{f,v}$ (for all finite dimensional (type **1**) U-modules V and all f and v) is closed under multiplication. Since the unit of U^* is $\epsilon = c_{1^*,1}$, where $\Bbbk \cdot 1$ is the trivial U-module, the subspace is a unital subalgebra of U^*.

In the classical case, we know that $\Bbbk[G]$ is a Hopf algebra with comultiplication $\Delta = m^*$, where m is the multiplication map $G \times G \to G$. But we defined $\Bbbk_q[G]$ in terms of matrix coefficients so it is not completely clear that this ring is a Hopf algebra.

In the finite dimensional setting, the dual of a Hopf algebra is a Hopf algebra. But in our case the natural embedding $U^* \otimes U^* \longrightarrow (U \otimes U)^*$ is not surjective so we cannot use $\mu : U \otimes U \longrightarrow U$ to define a comultiplication $\mu^* : U^* \longrightarrow (U \otimes U)^*$ to make U^* a Hopf algebra. But if restricting μ^* to $\Bbbk_q[G]$ we have image in $\Bbbk_q[G] \otimes \Bbbk_q[G]$, then $\Bbbk_q[G]$ can be made a Hopf algebra with comultiplication $\Delta = \mu^*$.

Lemma 3.4. *Let $\mu : U \otimes U \to U$ be the multiplication map and $\mu^* : \Bbbk_q[G] \to (U \otimes U)^*$. Then, $\mu^*(\Bbbk_q[G]) \subset \Bbbk_q[G] \otimes \Bbbk_q[G]$.*

Proof. Since $\Bbbk_q[G]$ is spanned by matrix coefficients, it suffices to prove that for $v \in V$ and $f \in V^*$, $m^*(c_{f,v}) \in \Bbbk_q[G] \otimes \Bbbk_q[G]$. To show this, let $v_1, ..., v_n$ and $f_1, ..., f_n$ be dual bases for V and V^*. Then for $u, u' \in U$ we have

$$\begin{aligned}
\mu^*(c_{f,v})(u \otimes u') &= f(uu' \cdot v) \\
&= f(u \sum f_i(u'v)v_i) \\
&= \sum f(uv_i) f_i(u'v) \\
&= \sum c_{f,v_i}(u) c_{f_i,v}(u') \\
&= \sum c_{f,v_i} \otimes c_{f_i,v}(u \otimes u').
\end{aligned}$$

□

The counit $\epsilon : \Bbbk_q[G] \to \Bbbk$ is defined by restricting the dual of the unit $\eta : \Bbbk \to U^*$; it satisfies $\epsilon(c_{f,v}) = c_{f,v}(1) = f(v)$. The antipode $S : \Bbbk_q[G] \to \Bbbk_q[G]$ is the restriction of $\lambda \in U^* \mapsto \lambda \circ S \in U^*$; this map satisfies $c_{f,v} \mapsto c_{v^{**},f}$, where $v^*{*}$ is the image of v under the vector space isomorphism $V \to V^*{*}$.

3.3. **Relations in $\Bbbk_q[G]$ from the R matrix.** Let V be a finite dimensional U module with basis v_i and dual basis v_i^* (keep in mind V^* is a U-module). We write

$$c_{ij} = c_{v_i^*, v_j} \tag{3.11}$$

so

$$u \cdot v_j = \sum c_{ij}(u) v_i \tag{3.12}$$

Let W be another finite dimensional U module with basis w_i, dual basis w_j^* and cmatrix coefficients $d_{ij} = d_{w_i^*, w_j}$.

Given an isomorphism $R : W \otimes V \longrightarrow V \otimes W$ we write

$$R(w_i \otimes v_j) = \sum_{h,l} R_{ij}^{hl} v_h \otimes w_l. \tag{3.13}$$

Lemma 3.5. *The following relation holds in $\Bbbk_q[G]$*

$$\sum_{h,l} R_{hl}^{rs} d_{li} c_{hj} = \sum_{h,l} R_{ij}^{hl} c_{rh} d_{sl} \tag{3.14}$$

Proof. Since $c_{w_i^* \otimes v_j^*, w_l \otimes v_h} = c_{w_i^*, w_l} c_{v_j^*, v_h}$, we find

$$u \cdot (w_i \otimes v_j) = \sum_{h,l} (d_{li} c_{hj})(u) w_l \otimes v_h. \tag{3.15}$$

The result follows from expanding both sides of

$$R(u(w_i \otimes v_j)) = u(R(w_i \otimes v_j)). \tag{3.16}$$

and comparing coefficients.

□

3.4. **Example of $\Bbbk_q[SL_2]$.** We computed R explicitly in the case of $\mathfrak{g} = \mathfrak{sl}_2$ and $V = W = L(\varpi_1)$. If we relabel our basis by $v_1 = v$ and $v_2 = Fv$, and write $v = q^{\frac{1}{2}}$, we get:

$$\begin{pmatrix} R_{11}^{11} & R_{11}^{12} & R_{12}^{11} & R_{12}^{12} \\ R_{11}^{21} & R_{11}^{22} & R_{12}^{21} & R_{12}^{22} \\ R_{21}^{11} & R_{21}^{12} & R_{22}^{11} & R_{22}^{12} \\ R_{21}^{21} & R_{21}^{22} & R_{22}^{21} & R_{22}^{22} \end{pmatrix} = \begin{pmatrix} v^{-1} & 0 & 0 & 0 \\ 0 & 0 & v & 0 \\ 0 & v & v^{-1} - v^3 & 0 \\ 0 & 0 & 0 & v^{-1} \end{pmatrix} \tag{3.17}$$

After writing

$$C = \begin{pmatrix} c_{11}c_{11} & c_{11}c_{12} & c_{12}c_{11} & c_{12}c_{12} \\ c_{11}c_{21} & c_{11}c_{22} & c_{12}c_{21} & c_{12}c_{22} \\ c_{21}c_{11} & c_{21}c_{12} & c_{22}c_{11} & c_{22}c_{12} \\ c_{21}c_{21} & c_{21}c_{22} & c_{22}c_{21} & c_{22}c_{22}, \end{pmatrix} \tag{3.18}$$

the relations in $\Bbbk_q[SL_2]$ given by R can be read off the equation $RC = CR$. Explicitly, we find the following relations hold (note that $v^2 = q$):

$$c_{11}c_{12} = v^2 c_{12}c_{11}, c_{11}c_{21} = v^2 c_{21}c_{11}, c_{12}c_{22} = v^2 c_{22}c_{12}, c_{21}c_{22} = v^2 c_{22}c_{21} \tag{3.19}$$

$$c_{12}c_{21} = c_{21}c_{21} \tag{3.20}$$

and

$$c_{11}c_{22} - c_{22}c_{11} = (v^2 - v^{-2}) c_{12}c_{21}. \tag{3.21}$$

Since all simple modules are direct summands of tensor products of $L(\varpi_1)$ all the matrix coefficients are expressed in terms of the c_{ij}'s. In other words, the matrix coefficients c_{ij} already generate $\Bbbk_q[SL_2]$ as an algebra.

There is one further relation in $\Bbbk_q[SL_2]$ which is the quantum determinant is identically equal to 1

$$c_{11}c_{22} - v^2 c_{12}c_{21} = 1. \tag{3.22}$$

and it turns out this is then a complete set of relations for $\Bbbk_q[SL_2]$.

References

[1] Alexander Bakalov, Bojko; Kirillov Jr. *Lectures on Tensor Categories and Modular Functors*, volume 21 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2001. 11

[2] Jens Carsten Jantzen. *Lectures on Quantum Groups*, volume 6 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, first edition, 1996. 4, 7, 12

[3] Jens Carsten Jantzen. *Representations of algebraic groups*, volume 107 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, second edition, 2003. 15