Equivariant K-theory and affine Hecke algebra.

Work over $\mathbb{C}$.

$$G\times G \xrightarrow{m} G$$

X - variety (not nec. smooth)

G - affine algebraic group acting on X.

$$G\times X \underset{a}{\overset{p}{\rightrightarrows}} X$$

A G-equivariant sheaf on X is a quasicoherent sheaf $\mathcal{F}$ together with an isomorphism $a^*\mathcal{F} \xrightarrow{\sim} p^*\mathcal{F}$ satisfying on $G\times G\times X$:

$$\begin{tikzcd}
(id\times a)^* a^*\mathcal{F} \arrow[r] \arrow[d, equal] & (id\times a)^* p^*\mathcal{F} = p_{23}^* a^*\mathcal{F} \arrow[r] & p_{23}^* p^*\mathcal{F} \arrow[d, equal] \\
(m\times id)^* a^*\mathcal{F} \arrow[r] & (m\times id)^* p^*\mathcal{F} = & p_{23}^* p^*\mathcal{F}.
\end{tikzcd}$$

Equivariant structure on $\mathcal{F}$ can (and should!) be thought of as giving a way to identify $\mathcal{F}(U)$ with $\mathcal{F}(gU)$ "$f \mapsto f\circ g^{-1}$" which ~~satisfies~~ is compatible with the group structure and the algebraic structure on G.

Actually, we will work only with coherent sheaves. Category $Coh^G(X)$.

Examples • A canonically defined sheaf - e.g. $\mathcal{O}_X, \Omega_X, T_X$

• If G acts transitively, fix $x\in X$ closed pt. and $H = Stab_G\, x$. Then one easily sees that a G-equivariant sheaf on X is uniquely determined by its fiber at x, which is a representation of H. Indeed given a representation V of H we construct the appropriate G-equivariant sheaf on $X = G/H$ by 'descent': start with trivial sheaf $G\times V$ and quotient by the action of H, $h(g,v) = (gh^{-1}, hv)$.

- G-equivariant sheaves on a point are f.d. representations of G. (In general case they are <u>algebraic</u> representations of G - but we work in coherent case, this is automatic).

- More generally if G acts trivially, one checks that $\mathrm{Coh}^G(X) = \mathrm{Rep}(G) \otimes \mathrm{Coh}(X)$. (essentially by Schur's lemma).

- Back to earlier example: if $X = \mathcal{B} = G/B$ flag variety, write L_λ for the line bundle associated to $\mathbb{C}_\lambda$ (λ an integral weight, G semisimple simply connected).
Then L_λ is also described as $L_\lambda(U) = \{ f \in \mathcal{O}_G(\pi^{-1}U) \text{ s.t. } f(gb) = \lambda^{-1}(b) f(g) \}$,
Since this is geometry, we should negate any arithmetic data. So we take the $\mathfrak{h}$-weights of $\mathfrak{n}$ to be the <u>negative</u> roots; thus for λ any <u>antidominant</u> integral weight we have the f.d. irrep V_λ, natural map $f : \mathcal{B} \longrightarrow \mathbb{P}V_\lambda$ (with image the unique closed G-orbit) and one observes that $f^*(\mathcal{O}(1)) = L_{-\lambda} = L_\lambda^*$.
So for λ dominant, L_λ is generated by its global sections, which as a G-rep is just $V_{w_0\lambda}$. (For λ also regular, L_λ is ample). This is Borel-Weil.

- $f : X \to Y$ G-equivariant, then f^*, f_* preserve G-equivariance (i.e. if $\mathcal{F}$ is given a G-equivariant structure, then $f^*\mathcal{F}, f_*\mathcal{F}$ have canonical G-equivariant structures).
Moreover: the Godement (flasque) resolution is naturally G-equivariant, and it is known that flat G-equivariant resolutions exist, so Lf^*, Rf_* also preserve G-equivariance.

__K-theory__ $\mathcal{C}$ an abelian category. There is a simplicial complex $B^+\mathcal{C}$, and the K-groups of $\mathcal{C}$ are given by $K_i(\mathcal{C}) = \pi_i(B^+\mathcal{C})$. I won't go into details, but it is known that $K_0(\mathcal{C}) = K(\mathcal{C})$, the Grothendieck group. This is the main object of study. We will take $\mathcal{C} = \mathrm{Coh}^G(X)$.

We still care about $K_i(\mathcal{C})$, because if $\mathcal{D}$ is a quotient of $\mathcal{C}$ by $\mathcal{E}$, we have the l.e.s.

~~Operations on K-theory~~ $\dots \to K_i(\mathcal{E}) \to K_i(\mathcal{C}) \to K_i(\mathcal{D}) \to K_{i-1}(\mathcal{E}) \to \dots$

$f: X \to Y$ G-equivariant, can we construct a pullback map $f^*: K^G(Y) \to K^G(X)$?

The trouble is that f^* is not exact, so if $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ on Y, it means that $[\mathcal{F}] + [\mathcal{H}] = [\mathcal{G}]$ in $K^G(Y)$; but $0 \to f^*\mathcal{F} \to f^*\mathcal{G} \to f^*\mathcal{H} \to 0$ may not be exact, so that $[f^*\mathcal{F}] + [f^*\mathcal{H}] \neq [f^*\mathcal{G}]$.

But from the long exact sequence

$$\dots \to L^1 f^*\mathcal{H} \to f^*\mathcal{F} \to f^*\mathcal{G} \to f^*\mathcal{H} \to 0$$

we see that, in $K^G(X)$,
$$\sum_{i=0}^{\infty} (-1)^i L^i f^*\mathcal{F} + \sum_{i=0}^{\infty} (-1)^i L^i f^*\mathcal{H} = \sum_{i=0}^{\infty} (-1)^i L^i f^*\mathcal{G},$$

if these expressions make sense.
(i.e. $L^i f^*\underset{\mathcal{G},\mathcal{H}}{\mathcal{F}} = 0 \quad \forall i \gg 0$).

We will consider two main cases:

- f is flat – then $L^i f^* = 0 \quad \forall i > 0$
- f is the closed embedding between smooth varieties

so that $Lf^* = \overset{L}{\otimes}\, \mathcal{O}_X$ and it is known that $\mathcal{O}_X$ has a locally free G-equivariant resolution of length $\leq \dim Y$.

(maybe $\leq \operatorname{codim} X$?)

We also have a way to deal with singular varieties. Take X, Y as before $X \overset{f}{\hookrightarrow} Y$ and Z a singular closed subvariety of Y.

$$\begin{tikzcd} X \cap Z \arrow[r, hook] \arrow[d, "f"] \arrow[dr, phantom, "\square"] & Z \arrow[d, "f i"] \\ X \arrow[r, hook, "f"] & Y \end{tikzcd}$$

Since pushforward along closed embedding is exact, we have $K^G(Z) \subset K^G(Y)$ a (non-unital) subalgebra. (i.e. with a different unit)

On the level of sheaves, $F \in Coh^G(Z)$ the thing to do is consider

$$\sum_{i=0}^{\infty} (-1)^i L^i f^* i_* F = \sum_{i=0}^{\infty} (-1)^i \mathrm{Tor}^i_{\mathcal{O}_Y}(f_* \mathcal{O}_X, i_* F)$$

Each $\mathrm{Tor}^i_{\mathcal{O}_Y}(f_* \mathcal{O}_X, i_* F)$ is supported ~~scheme-theoretically~~ ~~scheme~~ at least ~~scheme~~ set-theoretically on $X \cap Z$, indeed it is supported scheme-theoretically on both X and Z. So $I^n_{X \cap Z} \mathrm{Tor}^i_{\mathcal{O}_Y}(-) = 0$ some $n > 0$

Thus we have a filtration

$$\mathrm{Tor}^i_{\mathcal{O}_Y}(-) = \mathrm{Ann}_{\mathrm{Tor}^i_{\mathcal{O}_Y}(-)}(I^n_{X \cap Z}) \supset \mathrm{Ann}_{\dots}(I^{n-1}_{X \cap Z}) \supset \dots \supset 0$$

whose subquotients are all supported scheme-theoretically on $X \cap Z$. Thus $\mathrm{Tor}^i_{\mathcal{O}_Y}(f_* \mathcal{O}_X, i_* F)$ may be viewed as an element of $K^G(X \cap Z)$ by replacing it with the sum of all its ~~quotie~~ subquotients.

Tensor product may now be defined as $\Delta^* \circ \boxtimes$.

Tensoring with a vector bundle (flat coherent) can be done naively.

We can similarly define pushforward on K^G. This time, singularity is not an issue. We will assume f is proper, so that the higher derived pushforward groups of coherent sheaves eventually vanish.

<u>Convolution</u> X_1, X_2, X_3 smooth

$$Z_{12} \hookrightarrow X_1 \times X_2, \quad Z_{23} \hookrightarrow X_2 \times X_3.$$

Consider $Z = Z_{12} \underset{X_2}{\times} X_{23} \hookrightarrow X_1 \times X_2 \times X_3$

and assume $p_{13}|_Z$ is proper; then the convolution

$$K^G(Z_{12}) \otimes K^G(Z_{23}) \longrightarrow K^G(Z_{12} \circ Z_{23}) \quad (Z_{12} \circ Z_{23} := p_{13}(Z))$$

is given by $F \otimes G \longmapsto p_{13}|_Z {}_* (p_{12}^* F \otimes p_{23}^* G)$.

<u>Remark</u> If X is smooth then the two maps

$$K^G(X_\Delta) \otimes K^G(X_\Delta) \longrightarrow K^G(X_\Delta)$$

(convolution and tensor product) are equal.

We are now (sort of) ready to state the theorem.

Consider the nilpotent cone $\mathcal{N}$, the springer resolution

$$\tilde{\mathcal{N}} = T^*\mathcal{B} \longrightarrow \mathcal{N} \qquad \tilde{\mathcal{N}} = \{ (x, b) \in \mathcal{N} \times \mathcal{B} \mid x \in b \}$$

and the Steinberg variety $Z = \tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}} \subset \tilde{\mathcal{N}} \times \tilde{\mathcal{N}} = T^*(\mathcal{B} \times \mathcal{B})$

$\tilde{\mathcal{N}}$ is smooth. $Z \circ Z = Z$. $Z_\Delta := \tilde{\mathcal{N}}_\Delta \hookrightarrow Z$.

We view these varieties as $G \times \mathbb{C}^*$ representations: G acts as usual, while $\mathbb{C}^*$ acts as dilation on $\mathcal{N}$, and all the maps are G-equivariant (so $\mathbb{C}^*$ acts as dilation on the fiber of $T^*\mathcal{B} \to \mathcal{B}$, and trivially on $\mathcal{B}$).

Then $K^{G \times \mathbb{C}^*}(Z)$ has an algebra structure under convolution. It is of course naturally a $\mathrm{Rep}(G \times \mathbb{C}^*) = \mathrm{Rep}(G) \otimes \mathbb{Z}[q, q^{-1}]$

$$= \mathbb{Z}[P]^W \otimes \mathbb{Z}[q, q^{-1}]$$
$$= \mathrm{Rep}(T)^W \otimes \mathbb{Z}[q, q^{-1}]$$

-module

We claim that it is isomorphic to H the affine Hecke algebra.

(reminder: $(T_s+1)(T_s-q)=0$, $T_y T_w = T_{yw}$ for $l(yw) = l(y) + l(w)$;

$s_\alpha \lambda = \lambda \Rightarrow T_s e^\lambda = e^\lambda T_s$

$s_\alpha \lambda = \lambda - \alpha \Rightarrow T_{s_\alpha} e^{s_\alpha(\lambda)} T_{s_\alpha} = q e^\lambda$

+ more generally $T_{s_\alpha} e^{s_\alpha(\lambda)} - e^\lambda T_{s_\alpha} = (1-q) \dfrac{e^\lambda - e^{s_\alpha(\lambda)}}{1 - e^{-\alpha}}$).

Moreover, we have an isomorphism $K^{G \times \mathbb{C}^*}(Z_\Delta) \xrightarrow{\sim} \mathrm{Rep}[T][q, q^{-1}]$

which I will shortly explain, making the diagram

$$\begin{array}{ccc} K^{G\times\mathbb{C}^*}(Z_\Delta) & \longrightarrow & K^{G\times\mathbb{C}^*}(Z) \\ \Big\downarrow\wr & & \Big\downarrow\wr \\ \mathrm{Rep}[T][q,q^{-1}] & \longrightarrow & H \end{array}$$

commute.

How to see $K^{G\times\mathbb{C}^*}(Z_\Delta) \cong \mathrm{Rep}[T][q,q^{-1}]$?

Z_Δ is the conormal bundle to $\mathcal{B}_\Delta$ in $\mathcal{B}\times\mathcal{B}$

$$\begin{aligned} K^{G\times\mathbb{C}^*}(Z_\Delta) &= K^{G\times\mathbb{C}^*}\left(T^*_{\mathcal{B}_\Delta}(\mathcal{B}\times\mathcal{B})\right) \\ &\cong K^{G\times\mathbb{C}^*}(\mathcal{B}_\Delta) \\ &\cong K^G(\mathcal{B}_\Delta)\otimes \mathrm{Rep}(\mathbb{C}^*) \\ &\cong K^B(pt)\otimes \mathbb{Z}[q,q^{-1}] \\ &= \mathrm{Rep}[T][q,q^{-1}]. \end{aligned}$$

by the Thom isomorphism theorem. I come to it later.

It says if $\pi: E\to X$ is an affine fibration (e.g. vector bundle) then π^* is an isom. on K^G_j, all j. If E is a vector bundle w/ zero section i, then i^* is its inverse

Hopefully I will get to prove the theorem next week.

Let me prove a preliminary result.

<u>Proposition</u> ~~...~~ $K^{G\times\mathbb{C}^*}(Z)$ is free, of rank $|W|$, as a $K^{G\times\mathbb{C}^*}(\mathcal{B})$-module.

<u>Proof</u> We have $K^{G\times\mathbb{C}^*}(\mathcal{B}) = \mathrm{Rep}(T)[q,q^{-1}]$ by earlier results.
So by cellular fibration theorem (a slight generalization of Thom isomorphism theorem) it suffices to show that Z is a cellular fibration over $\mathcal{B}$, with $|W|$ cells.

Recall $\quad Z = \tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}} = T^*\mathcal{B} \times_{\mathcal{N}} T^*\mathcal{B} \hookrightarrow T^*\mathcal{B}\times T^*\mathcal{B}$

and $T^*\mathcal{B}\times T^*\mathcal{B} \cong T^*(\mathcal{B}\times\mathcal{B})$

This last isom. involves a sign

$$((x,\mathfrak{b}),(x',\mathfrak{b}')) \longmapsto (x\oplus -x', (\mathfrak{b},\mathfrak{b}'))$$

for technical reasons concerning the symplectic structure on T^*.

The tangent space at $(\mathfrak{b},\mathfrak{b}')$ to ~~the ... orbit~~ $\mathcal{B}\times\mathcal{B}$ is $\mathfrak{g}/\mathfrak{b}\oplus\mathfrak{g}/\mathfrak{b}'$

" " " " to the G-orbit in $\mathcal{B}\times\mathcal{B}$ through that point is thus the image of $\mathfrak{g}$ in $\mathfrak{g}/\mathfrak{b}\oplus\mathfrak{g}/\mathfrak{b}'$ (diagonal map). So the conormal space at that point is the annihilator (under the Killing form) of the ~~...~~ diagonal of $\subset \mathfrak{g}\oplus\mathfrak{g}$ of ~~...~~ $(\mathfrak{g}/\mathfrak{b})^*\oplus(\mathfrak{g}/\mathfrak{b}')^* \cong \mathfrak{n}\oplus\mathfrak{n}'$, i.e.

$\{$ ~~...~~ $x\oplus y\in\mathfrak{n}\oplus\mathfrak{n}' \mid (x+y,u)=0\ \forall u\in\mathfrak{g}\} = \{(x,-x)\in\mathfrak{n}\oplus\mathfrak{n}'\}$.

This is exactly the fiber above $(\mathfrak{b},\mathfrak{b}')$ of $Z\longrightarrow\mathcal{B}\times\mathcal{B}$.

This gives an alternate description of Z as the union to the conormal bundles of the (diagonal) G-orbits in $\mathcal{B}\times\mathcal{B}$.

These are parameterised by W, since every G-orbit contains a unique element of the form (B, wB) $\quad w\in W$.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cong (b, b_w)$

Write Y_w for this orbit; so $\mathcal{B}\times\mathcal{B} = \coprod_{w\in W} Y_w$ and

$$Z = \coprod_{w\in W} \underbrace{T^*_{Y_w}(\mathcal{B}\times\mathcal{B})}_{=Z_w}$$

Consider the ~~second~~ first projection $\mathcal{B}\times\mathcal{B} \to \mathcal{B}$

~~The fiber of b_g~~ It gives by restriction maps $Y_w \to \mathcal{B}$, in which the fiber of b_g is ~~isomorphic to~~ $\{b_g\}\times \mathrm{Ad}_g(\mathcal{B}_w) \cong \mathcal{B}_w$

($\mathcal{B}_w$ the Bruhat cell containing wB)

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \cong b_w$

Diagram

$$\begin{tikzcd}
\{b_g\}\times g\mathcal{B}_w \arrow[r, hook] \arrow[d, phantom, "\cap"] \arrow[dr, phantom, "\square"] & Y_w \arrow[d, phantom, "\cap"] \\
\{b_g\}\times \mathcal{B} \arrow[r, hook] & \mathcal{B}\times\mathcal{B}
\end{tikzcd}$$

$\Rightarrow$ fiber over $\{b_g\}$ of $T^*_{Y_w}(\mathcal{B}\times\mathcal{B})$ is $\cong T^*_{\mathcal{B}_w}\mathcal{B}$.

We see that $Y_w \to \mathcal{B}$ is an affine fibration (recall $\mathcal{B}_w$ is affine - in fact it is canonically a vectorspace with base point b_w).

Each Y_w is locally closed, and G-invariant, and its

$\quad Z_w \qquad\qquad\qquad\qquad\qquad\qquad\qquad G\times\mathbb{C}^*$-invariant closure is seen to be $Y_w \cup$ some $Y_{w'}$ of lower dimension

$$\subset Z_w \cup \overset{\text{some}}{\cancel{\text{...}}}\ Z_{w'}\text{, same } w'\text{s as above.}$$

Let $w_1, \dots, w_n$ enumerate W $(n = |W|)$ in such a way that $\dim Y_{w_i} \geq \dim Y_{w_{i+1}}$ $\forall i$. Then

$$Z^i : \bigcup_{j \geq i} Z_{w_j} \text{ is a closed } G \times \mathbb{C}^*\text{-invariant.}$$

So $Z = Z^1 \supset \dots \supset Z^n$ are fibrations over $\mathcal{B}$ and

$Z^i \setminus Z^{i+1} = Z_{w_i}$ is affine over $\mathcal{B}$.

We are able to deduce (cellular fibration lemma) that $K^{G \times \mathbb{C}^*}(Z)$ is free of rank $|W|$ over $K^{G\times\mathbb{C}^*}(\mathcal{B})$.

(So we haven't disproved that it is the Hecke algebra!).

We should discuss <u>Thom isom. thm</u>. First a few aside.

$\pi : V \to X$ a G-equivariant vector bundle, $i : X \to V$ the zero section.

π being flat, π^* is easy to handle. Would like to understand i^*.

It is the same as tensoring with $i_* \mathcal{O}_X$, so we need a flat resolution ~~of~~ of $i_* \mathcal{O}_X$. It is called the Koszul complex

Consider surjection $\mathcal{O}_V \to i_* \mathcal{O}_X$. What is its kernel?

Fibrewise, it is functions on the vector space V_x vanishing at 0, namely the ideal of ~~$\cdot$~~ $\mathrm{Sym}\, V_x^*$ generated by V_x^*. This construction glues over the fibers, and we see that the map

$$\pi^* V^\vee \longrightarrow \mathcal{O}_V$$

fibres $(\mathrm{Sym}\, V_x^*) \otimes V_x^*$

$$f \otimes \theta \longmapsto f\theta \qquad \text{surjects to the kernel.}$$

I leave it as ~~an~~ exercise that a similar procedure allows to define

$$\cdots \to \pi^*(\Lambda^2 V^\vee) \to \pi^*(\Lambda^1 V^\vee) \to \underset{\substack{\parallel \\ \mathcal{O}_V}}{\pi^*(\Lambda^0 V^\vee)} \to 0.$$

which is a flat complex quasiisomorphic to $i_* \mathcal{O}_X$.

<u>Remark</u> $\pi^*(\Lambda^j V^\vee) \cong \Omega^j_{V/X}$.

Let $\lambda(V) = \sum_{i=0}^{\dim V} (-1)^i [\Lambda^i V] = \sum_{i=0}^{\infty} (-1)^i [\Lambda^i V]. \in K^G(X)$

So $\underset{\substack{\parallel \\ i_*[\mathcal{O}_X]}}{[i_* \mathcal{O}_X]} = \pi^* \lambda(V^\vee)$.

Now observe that $i^* \pi^* = \mathrm{id}$ and $i^* i_*$ is given by multiplication (tensoring) by $\lambda(V^\vee)$.

<u>Remark</u> Using a (very clever) algebraic analogue of tubular nbhd ~~theorem~~, one can show that if $i : N \hookrightarrow M$ of smooth varieties, then $i^* i_*$ is given by multiplication by $\lambda(T^*_N M)$.

An example Let $\mathbb{P}$ be a projective space $\mathbb{P} = \mathbb{P}^n$ some n, $= \mathbb{P}(V)$

$\mathcal{O}_\Delta = \Delta_* \mathcal{O}_{\mathbb{P}}$ be the structure sheaf of the diagonal in $\mathbb{P} \times \mathbb{P}$.

It has a resolution, the Beilinson resolution.

$$\mathcal{O}_{\mathbb{P}}(-n) \boxtimes \Omega^n_{\mathbb{P}}(n) \to \cdots \to \mathcal{O}_{\mathbb{P}}(-1) \boxtimes \Omega^1_{\mathbb{P}}(1) \to \mathcal{O}_{\mathbb{P}\times\mathbb{P}} \to \mathcal{O}_\Delta \to 0$$

Idea of construction: start w/ $0 \to \mathcal{O}_{\mathbb{P}} \to V \otimes \mathcal{O}_{\mathbb{P}}(1) \to T_{\mathbb{P}} \to 0$

(fibrewise: $0 \to \mathbb{C} \to \mathrm{Hom}(\ell, V) \to \mathrm{Hom}(\ell, V/\ell) \to 0$).

$\rightsquigarrow 0 \to \mathcal{O}_{\mathbb{P}}(-1) \to V \otimes \mathcal{O}_{\mathbb{P}} \to Q_{\mathbb{P}} \to 0$

$\Omega^1_{\mathbb{P}}(1) = Q_{\mathbb{P}}^\vee$ embeds in $V^* \otimes \mathcal{O}_{\mathbb{P}}$ w/ cokernel $\mathcal{O}_{\mathbb{P}}(1)$.

So the map $\mathcal{O}_{\mathbb{P}}(-1) \boxtimes \Omega^1_{\mathbb{P}}(1) \to \mathcal{O}_{\mathbb{P}\times\mathbb{P}}$ is defined as

$$\mathcal{O}_{\mathbb{P}}(-1) \boxtimes \Omega^1_{\mathbb{P}}(1) \hookrightarrow \mathcal{O}_{\mathbb{P}}(-1) \boxtimes V^* \otimes \mathcal{O}_{\mathbb{P}} = V^* \otimes \mathcal{O}_{\mathbb{P}}(-1) \boxtimes \mathcal{O}_{\mathbb{P}}$$
$$\downarrow$$
$$\mathcal{O}_{\mathbb{P}} \boxtimes \mathcal{O}_{\mathbb{P}} = \mathcal{O}_{\mathbb{P}\times\mathbb{P}}$$

by the natural map $V^* \otimes \mathcal{O}_{\mathbb{P}}(-1) \to \mathcal{O}_{\mathbb{P}}(1) \otimes \mathcal{O}_{\mathbb{P}}(-1) = \mathcal{O}_{\mathbb{P}}$.

Fiberwise this is $\ell' \otimes \mathrm{Ann}_{V^*} \ell \to \mathbb{C}$. ✓

Then use Λ^k: $\mathcal{O}_{\mathbb{P}}(-1) \boxtimes \Omega^1_{\mathbb{P}}(1) = \mathcal{O}_{\mathbb{P}}(-k) \boxtimes \Omega^k_{\mathbb{P}}(k)$

Now suppose $E \to X$ a G-equivariant vector bundle
Consider $\mathbb{P} = \mathbb{P}(E)$ the projective bundle over X. $\pi : \mathbb{P} \to X$
Beilinson resolution naturally glues: get

$$\cdots \to \mathcal{O}_{\mathbb{P}}(-1) \boxtimes \Omega^1_{\mathbb{P}/X}(1) \to \mathcal{O}_{\mathbb{P} \times_X \mathbb{P}} \to \mathcal{O}_\Delta \to 0 .$$

relative differentials

$\mathcal{O}_{\mathbb{P}}(k)$ is fibrewise $\mathcal{O}_{\mathbb{P}}(k)$

This is a G-equivariant resolution!
Consider following diagram.

$$\begin{tikzcd}
\mathbb{P} \times_X \mathbb{P} \arrow[rr, "p_2"] \arrow[dd, "p_1"'] & & \mathbb{P} \arrow[dd, "\pi"] \\
& \mathbb{P}_\Delta \arrow[ul, "\Delta"] \arrow[ur, "p'_2"'] \arrow[dl, "p'_2"'] \arrow[dr, "\pi"] & \\
\mathbb{P} \arrow[rr, "\pi"'] & & X
\end{tikzcd}$$

Outer diagram is Cartesian.
π ~~[illegible]~~ flat - will use flat base change.

Let $\mathcal{F} \in \mathrm{Coh}^G(\mathbb{P})$

Consider $p_{1*}(\mathcal{O}_\Delta \otimes p_2^* \mathcal{F}) = p_{1*}(\Delta_* \mathcal{O}_{\mathbb{P}_\Delta} \otimes p_2^* \mathcal{F})$

$$= p_{1*} \Delta_* (\mathcal{O}_{\mathbb{P}_\Delta} \otimes \cancel{\Delta^* p_2^*} \mathcal{F})$$

$$= \mathcal{F}$$

$$= p_{1*}\left(\sum_{i=0}^{n} (-1)^i \mathcal{O}_{\mathbb{P}}(-i) \boxtimes \Omega^i_{\mathbb{P}/X}(i) \otimes p_2^* \mathcal{F} \right)$$

$$= \sum_{i=0}^{n} (-1)^i p_{1*}\left(p_1^* \mathcal{O}_{\mathbb{P}}(-i) \otimes p_2^*(\Omega^i_{\mathbb{P}/X}(i) \otimes \mathcal{F}) \right)$$

$$= \sum_{i=0}^{n} (-1)^i \mathcal{O}_{\mathbb{P}}(-i) \otimes \underbrace{p_{1*} p_2^*}_{\pi^* \pi_*} (\Omega^i_{\mathbb{P}/X}(i) \otimes \mathcal{F}))$$

ok.

$\rightsquigarrow$ can write as span of $\mathcal{O}_{\mathbb{P}}, \ldots, \mathcal{O}_{\mathbb{P}}(n)$

Rks | Argument actually shows $\mathcal{F}$ is q-is. to complex with terms of form
Technique. $\rightsquigarrow$ can apply general results to get the same thing for $K_i^G(\mathbb{P})$

$\rightsquigarrow$ Resolution theorem!