CENTRAL ELEMENTS OF THE COMPLETED UNIVERSAL ENVELOPING ALGEBRA

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ABSTRACT. These notes were prepared form the Spring 2024 graduate learning seminar at MIT on representations of affine Kac–Moody algebras at the critical level. We closely follow Chapters 3.1-3.3 in [Fre07]. Our primary goal is to relate the vertex algebra associated with an affine Kac–Moody algebra with its completed universal enveloping algebra. We first discuss commutation relations for the Sugawara operators on the vacuum module for an affine Kac–Moody algebra. We aim not to focus on the computations, but rather the general framework surrounding these computations, and more importantly, how to recover information about the the universal enveloping algebra from these computations. Accordingly, we will then show that these relations also hold for the Sugawara elements in the completed universal enveloping algebra by considering the Lie algebra of Fourier coefficients. In particular, we will show that the Sugawara elements are central at the critical level. We continue our study with a discussion of the center of the vacuum vertex algebra for $\hat{\mathfrak{g}}$ and its relationship with the center of the completed universal enveloping algebra.

1. SUGAWARA OPERATORS

We begin by establishing some notational conventions. Let $\mathfrak{g}$ be a finite-dimensional simple complex Lie algebra, and let $\hat{\mathfrak{g}}_\kappa := \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}((t)) \oplus \mathbb{C}K$ be the corresponding affine Kac–Moody algebra, where κ is a choice of a nontrivial invariant bilinear form on $\mathfrak{g}$ (equivalently, this corresponds to choosing a nontrivial two-cocycle on $\mathfrak{g}((t))$). We will be particularly interested in the critical level $\kappa = \kappa_c := -\frac{1}{2}\kappa_{\mathfrak{g}}$, where $\kappa_{\mathfrak{g}}$ is the Killing form on $\mathfrak{g}$. Then, we let $U_\kappa(\hat{\mathfrak{g}}) := U(\hat{\mathfrak{g}}_\kappa)/(K-1)$ denote the quotient of the universal enveloping algebra of $\hat{\mathfrak{g}}_\kappa$ by the ideal generated by $K-1$. Write $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$ for its completion with respect to the descending filtration by the left ideals

$$J_N := U_\kappa(\hat{\mathfrak{g}}) \cdot \hat{\mathfrak{g}}(N),$$

where $\hat{\mathfrak{g}}(N) := \mathrm{Span}_{\mathbb{C}}\{x \otimes t^m \mid x \in \mathfrak{g},\ m \geq N\}$. For any $x \in \mathfrak{g}$ and $n \in \mathbb{Z}$, we write $x(n) := x \otimes t^n$. Let $V_\kappa(\mathfrak{g})$ denote the vacuum $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$-module. Recall that this module was defined as the induced module

$$U_\kappa(\hat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[[t]] \oplus \mathbb{C}K)} \mathbb{C}_\kappa,$$

where $\mathbb{C}_\kappa$ is the $U(\mathfrak{g}[[t]] \oplus \mathbb{C}K)$-module where $\mathfrak{g}[[t]]$ acts trivially and K acts by one.

We fix a nontrivial invariant bilinear form κ_0 on $\mathfrak{g}$. Then, after fixing some basis $x_1, \ldots, x_{\dim \mathfrak{g}}$ for $\mathfrak{g}$, we let $x^1, \ldots, x^{\dim \mathfrak{g}}$ denote the dual basis with respect to κ_0. We will write $X_i(z) = \sum_{n \in \mathbb{Z}} x_i(n) z^{-n-1}$ for all $i = 1, \ldots, \dim \mathfrak{g}$.

Definition 1.1. For each $n \in \mathbb{Z}$, we define the **Segal–Sugawara elements** (or more concisely, *Sugawara elements*)

$$S_n := \frac{1}{2} \sum_{i=1}^{\dim \mathfrak{g}} \sum_{m_1+m_2=n} : x_i(m_1) x^i(m_2) : \in \widetilde{U}_\kappa(\hat{\mathfrak{g}}).$$

Equivalently, these are the Fourier coefficients of the formal power series

$$\frac{1}{2} : X_i(z) X^i(z) : = \sum_{n \in \mathbb{Z}} S_n z^{-n-2}.$$

We refer to the images of the Sugawara elements in End $V_\kappa(\mathfrak{g})$ as the *Sugawara operators*, and we will reuse the notation S_n for their images as well. Similarly, we write $x_i(m)$ for the image of $x_i(m) \in \widetilde{U}_\kappa(\hat{\mathfrak{g}})$ in End $V_\kappa(\mathfrak{g})$ under the action map. Recall that our overarching goal is to study the center of $\tilde{U}_{\kappa_c}(\hat{\mathfrak{g}})$, and to do so, we will show that the Sugawara elements are indeed central at the critical level. We should think about the Sugawara elements as the infinite-dimensional analogue to the quadratic Casimir elements for finite-dimensional semisimple Lie algebras. In fact, in the case $\hat{\mathfrak{g}} = \hat{\mathfrak{sl}}_2$, these Sugawara elements actually generate the center, analogous to how the quadratic Casimir generates the center for $U(\mathfrak{sl}_2)$.

In this section, we will apply the vertex algebra formalism to compute commutation relations between the Sugawara *operators* and the endomorphisms $x_i(m)$ on $V_\kappa(\mathfrak{g})$. We want to emphasize that the vertex algebra formalism is not strictly necessary for performing these computations – for instance, in [KR87, Lecture 10], these relations are computed using completely elementary means. However, we should view the following computations as a basic blueprint to illustrate the general techniques underlying more difficult computations in the subsequent talks, when the vertex algebra formalism will become indispensable. Essentially, the advantage of working with the vertex algebra is due to its simpler structure – the completed universal enveloping algebra admits a filtration by degree, but the filtered pieces are infinite-dimensional and difficult to work with. On the other hand, the vertex algebra has a bona fide grading where each graded piece is finite-dimensional and thus, easier to understand. We will exploit this graded structure repeatedly in the following computations. Moreover, the completed universal enveloping algebra acts by endomorphisms on the vacuum module, so we may hope to translate our vertex algebra computations to results on $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$.

In what follows, we show that, at the critical level, the Sugawara operators are central in the image of $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$ under the action map. Moreover, we will see that away from the critical level, the Sugawara operators form a Lie subalgebra isomorphic to a quotient of the Virasoro algebra. This latter result will lead us to a brief digression on conformal vertex algebras. Unfortunately, the computations in this section will apply only in the endomorphism algebra of the vacuum module $V_\kappa(\mathfrak{g})$, and it is not obvious that they should also hold in the completed universal enveloping algebra. This will be the aim of the next section.

The following relation (see page 9 in Ilya's second set of notes) forms the cornerstone for all computations (read: exercises!) in this section. For any vertex algebra V and

$A, B \in V$, we have

$$[A_m, B_k] = \sum_{n\geq 0} \binom{m}{n} (A_n B)_{m+k-n}, \tag{1}$$

where we use the notation $Y(C,z) = \sum_{\ell\in\mathbb{Z}} C_\ell z^{-\ell-1}$ for any $C \in V$.

1.1. Commutation Relations for Sugawara Operators.

Proposition 1.2. [Fre07, Chapter 3.1.1] *For any $n, m \in \mathbb{Z}$ and $i = 1, \ldots, \dim \mathfrak{g}$, the following relation holds in* $\operatorname{End} V_\kappa(\mathfrak{g})$*:*

$$[S_n, x_i(m)] = \frac{\kappa_c - \kappa}{\kappa_0} \cdot m x_i(m+n),$$

where κ_0 is a fixed (nontrivial) invariant bilinear form on $\mathfrak{g}$ with respect to which $\{x_i\}$ and $\{x^i\}$ form a dual basis. In particular, at the critical level $\kappa = \kappa_c$, the operators S_k are central in the image of $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$ in $\operatorname{End} V_\kappa(\mathfrak{g})$.

Remark 1.3. Since the space of invariant forms on the simple Lie algebra $\mathfrak{g}$ is one-dimensional, the ratio $(\kappa_c - \kappa)/\kappa_0$ is a complex number.

Proof. Consider the vector

$$\sigma := \frac{1}{2} \sum_{i=1}^{\dim \mathfrak{g}} x_i(-1) x^i(-1) |0\rangle \in V_\kappa(\mathfrak{g}),$$

so the kth Sugawara operator S_n is the $(n+1)$st Fourier coefficient σ_{n+1} in the vertex operator

$$Y(\sigma, z) = \sum_{n\in\mathbb{Z}} \sigma_n z^{-n-1} = \frac{1}{2} \sum_{i=1}^{\dim \mathfrak{g}} : X_i(z) X^i(z) : .$$

We will compute the commutation relations for the Fourier coefficients of $Y(\sigma, z)$ and the Fourier coefficients of

$$X_i(z) = Y(x_i(-1)|0\rangle, z) = \sum_{m\in\mathbb{Z}} x_i(m) z^{-m-1}.$$

Consider the following special case of (1):

$$[x_i(m), \sigma_n] = \sum_{\ell\geq 0} \binom{m}{\ell} (x_i(\ell)\sigma)_{m+n-\ell}$$

for any $m, n \in \mathbb{Z}$. Hence, it suffices to compute the Fourier coefficients of $Y(x_i(\ell)\sigma, z)$ for each $\ell \geq 0$. By definition,

$$x_i(\ell)\sigma = \frac{1}{2} \sum_{j=1}^{\dim \mathfrak{g}} x_i(n) x_j(-1) x^j(-1) |0\rangle .$$

Observe that this vector is homogeneous of degree $n-2$, and since all nonzero homogeneous components of $V_\kappa(\mathfrak{g})$ have nonnegative degree, we have $x_i(\ell)\sigma = 0$ for all $\ell > 2$. Hence, it suffices to consider only three cases $\ell = 0, 1, 2$. (Here, we already see a slight advantage of the vertex algebra formalism!) In the interest of time, the rest of the proof will be left as an exercise during the talk. For a more direct approach without the vertex algebra theory, refer to [KR87, Lecture 10].

For completeness, here are the results of the cases.

$$x_i(0)\sigma = 0, \quad x_i(1)\sigma = \frac{\kappa - \kappa_c}{\kappa_0} x_i(-1)\,|0\rangle\,, \quad x_i(2)\sigma = 0.$$

Let us adopt Einstein summation notation for the remainder of this proof (so indices that appear as an upper and lower index in the same term are summed over).

Case 0. Suppose $\ell = 0$. Using the Lie bracket on $\hat{\mathfrak{g}}_\kappa$ and $x_i(0)\,|0\rangle = 0$, we have

$$x_i(0)\sigma = \frac{1}{2}([x_i, x_j](-1)x^j(-1) + x_j(-1)[x_i, x^j](-1))\,|0\rangle\,.$$

If we write $[x_i, x_j] = d_j^k x_k$, then we see that

$$d_j^k = \kappa_0([x_i, x_j], x^k) = -\kappa_0(x_j, [x_i, x^k]),$$

so that $[x_i, x^j] = -d_k^j x^k$. Hence,

$$x_i(0)\sigma = \frac{1}{2}(d_j^k x_k(-1)x^j(-1) - d_k^j x_j(-1)x^k(-1))\,|0\rangle = 0.$$

Case 1. Now, take $\ell = 1$. In this case, the relations in $\hat{\mathfrak{g}}_\kappa$ give us

$$\begin{aligned} x_i(1)\sigma &= \frac{1}{2}([x_i, x_j](0)x^j(-1)\,|0\rangle + x_j(-1)[x_i, x^j](0)\,|0\rangle) \\ &\quad + \frac{1}{2}(\kappa(x_i, x_j)x^j(-1)\,|0\rangle + \kappa(x_i, x^j)x_j(-1)\,|0\rangle) \\ &= \frac{1}{2}[x_i, x_j](0)x^j(-1)\,|0\rangle + \frac{1}{2}(\kappa(x_i, x_j)x^j(-1)\,|0\rangle) + \frac{1}{2}\frac{\kappa}{\kappa_0}x_i(-1)\,|0\rangle\,, \\ &= \frac{1}{2}[x_i, x_j](0)x^j(-1)\,|0\rangle + \frac{\kappa}{\kappa_0}x_i(-1)\,|0\rangle\,, \end{aligned}$$

where we used the fact that $\kappa(x_i, x^j) = \delta_{ij}\frac{\kappa}{\kappa_0}$ for the second equality and $\kappa(x_i, x_j)x^j = \frac{\kappa}{\kappa_0}x_i$ for the last equality. Now, using $[x_i, x_j](0)\,|0\rangle = 0$, let us rewrite

$$\frac{1}{2}[x_i, x_j](0)x^j(-1)\,|0\rangle = \frac{1}{2}[[x_i, x_j], x^j](-1)\,|0\rangle = \frac{1}{2}[x^j, [x_j, x_i]](-1)\,|0\rangle\,.$$

Note that $\mathfrak{g}$ acts on $\mathfrak{g}(-1)\,|0\rangle$ by the adjoint action, so the latter sum above is simply the action of the quadratic Casimir $\frac{1}{2}x^j x_j$ on $x_i(-1)\,|0\rangle$ via the adjoint action. The adjoint representation of $\mathfrak{g}$ is irreducible since $\mathfrak{g}$ is simple, so it follows that the action of the quadratic Casimir is by scalar multiplication by some $\lambda \in \mathbb{C}$, that is,

$$\frac{1}{2}[x_i, x_j](0)x^j(-1)\,|0\rangle = \lambda x_i(-1)\,|0\rangle\,.$$

Let $\kappa_{\mathfrak{g}}$ denote the Killing form on $\mathfrak{g}$. Then, the scalar by which $\frac{1}{2}x_j x^j$ acts on the adjoint representation is precisely

$$\lambda = \frac{1}{2\dim\mathfrak{g}}\operatorname{Tr}(\operatorname{ad}(x^j)\circ\operatorname{ad}(x_j)) = \frac{1}{2\dim\mathfrak{g}}\kappa_{\mathfrak{g}}(x^j, x_j) = \frac{1}{2}\frac{\kappa_g}{\kappa_0}.$$

Thus, we conclude that

$$x_i(1)\sigma = \left(\frac{(1/2)\kappa_{\mathfrak{g}} + \kappa}{\kappa_0}\right) x_i(-1)\,|0\rangle = \left(\frac{\kappa - \kappa_c}{\kappa_0}\right) x_i(-1)\,|0\rangle\,.$$

Case 2. Finally, take $\ell = 2$. In this case, the relations in $\hat{\mathfrak{g}}_\kappa$ give us

$$\begin{aligned} x_i(2)\sigma &= \frac{1}{2}([x_i, x_j](1)x^j(-1)\,|0\rangle + x_j(-1)[x_i, x^j](1)\,|0\rangle) \\ &= \frac{1}{2}[x_i, x_j](1)x^j(-1)\,|0\rangle = -\frac{1}{2}\kappa(x_i, [x_j, x^j])\,|0\rangle\,. \end{aligned}$$

If we assume without loss of generality that $\{x_j\}$ is orthonormal with respect to κ_0, then $x^j = x_j$ for all $j = 1, \dots, \dim\mathfrak{g}$ and hence the right-hand side is equal to zero. Note that we can make such an assumption since the Sugawara elements S_k do not depend on the choice of basis, and hence, the left-hand side of the equation above is also independent of the choice of basis (we can extend x_i to an orthonormal basis). Thus, the result is zero.

Finishing Touches. To conclude, the cases above give us

$$[x_i(m), \sigma_n] = \sum_{\ell\geq 0}\binom{m}{\ell}(x_i(\ell)\sigma)_{m+n-\ell} = \left(\frac{\kappa - \kappa_c}{\kappa_0}\right) m x_i(m+n-1).$$

Using the fact that $\sigma_{n+1} = S_n$, we conclude

$$[S_n, x_i(m)] = \left(\frac{\kappa_c - \kappa}{\kappa_0}\right) m x_i(m+n). \qquad \square$$

Away from the critical level, let us define the *normalized Sugawara elements*

$$L_n := \frac{\kappa_0}{\kappa - \kappa_c} S_n.$$

Observe that Proposition 1.2 can be rephrased as

$$[L_n, x_i(m)] = -m x_i(m+n). \tag{2}$$

that is, the adjoint action of L_n on $V_\kappa(\mathfrak{g})$ is the action of the derivation $-t^{n+1}\partial_t$. Recall that these derivations form a topological basis for the Witt algebra $\mathbb{C}((t))\partial_t$, so a natural question is whether the Sugawara operators form a Lie subalgebra of $\operatorname{End} V_\kappa(\mathfrak{g})$ isomorphic to the Witt algebra. As it turns out, we will show that the commutation relations between the normalized Sugawara operators are the same as the commutation relations not of the Witt algebra, but rather a quotient of its central extension, the Virasoro algebra.

Proposition 1.4. [Fre07, Chapter 3.1.2] *For any $n, m \in \mathbb{Z}$, the following relation holds in* $\operatorname{End} V_\kappa(\mathfrak{g})$*:*

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3 - m}{12}\delta_{m+n,0}c_\kappa,$$

where $c_\kappa = \dim \mathfrak{g} \cdot \frac{\kappa}{\kappa - \kappa_c}$.

Proof. Write $\bar{\sigma} = \frac{\kappa_0}{\kappa - \kappa_c}\sigma$, so that $L_n = \bar{\sigma}_{n+1}$. In this case, we need to compute the OPE

$$Y(\bar{\sigma}, z)Y(\bar{\sigma}, w) \sim \sum_{n \geq 0} \frac{Y(\bar{\sigma}_n \cdot \bar{\sigma}, w)}{(z-w)^{n+1}}.$$

Note $\deg \bar{\sigma}_n = \deg S_{n-1} = 1 - n$, so $\deg \bar{\sigma}_n \cdot \bar{\sigma} = 3 - n$. Hence, $\bar{\sigma}_n \cdot \bar{\sigma} = 0$ for all $n \geq 4$. We consider the following four cases.

Case 0. First consider $n = 0$. Observe that $\bar{\sigma}_0 |0\rangle = 0$, so it follows that

$$\bar{\sigma}_0 \cdot \bar{\sigma} = \frac{\kappa_0}{2(\kappa - \kappa_c)}([\bar{\sigma}_0, x_i(-1)]x^i(-1) + x_i(-1)[\bar{\sigma}_0, x^i(-1)]) |0\rangle$$

From (2), we see that the adjoint action of $\bar{\sigma}_0$ is the same as the action of the derivation $-\partial_t = T$, so it follows that

$$\bar{\sigma}_0 \cdot \bar{\sigma} = \frac{\kappa_0}{2(\kappa - \kappa_c)} - \partial_t(x_i(-1)x^i(-1)) |0\rangle = T\bar{\sigma}.$$

Case 1. Now suppose $n = 1$. In this case, (2) tells us that $[\bar{\sigma}_1, -]$ acts as the grading operator, so

$$\begin{aligned}\bar{\sigma}_1 \cdot \bar{\sigma} &= \frac{\kappa_0}{2(\kappa - \kappa_c)}([\bar{\sigma}_1, x_i(-1)]x^i(-1) + x_i(-1)[\bar{\sigma}_1, x^i(-1)]) |0\rangle \\ &= \frac{\kappa_0}{2(\kappa - \kappa_c)}(x_i(-1)x^i(-1) + x_i(-1)x^i(-1)) |0\rangle = 2\bar{\sigma}.\end{aligned}$$

Case 2. Now suppose $n = 2$. In this case, Equation (2) tells us that $[\bar{\sigma}_2, x(m)] = -mx(m+1)$ for any $x \in \mathfrak{g}$, so it follows that

$$\begin{aligned}\bar{\sigma}_2 \cdot \bar{\sigma} &= \frac{\kappa_0}{2(\kappa - \kappa_c)}([\bar{\sigma}_2, x_i(-1)]x^i(-1) + x_i(-1)[\bar{\sigma}_2, x^i(-1)]) |0\rangle \\ &= \frac{\kappa_0}{2(\kappa - \kappa_c)}(x_i(0)x^i(-1) + x_i(-1)x^i(0)) |0\rangle \\ &= \frac{\kappa_0}{2(\kappa - \kappa_c)}(x_i(0)x^i(-1)) |0\rangle \\ &= \frac{\kappa_0}{2(\kappa - \kappa_c)}[x_i, x^i](-1) |0\rangle.\end{aligned}$$

If we choose $\{x_i\}$ orthonormal with respect to κ_0 (note that this choice does not affect $\bar{\sigma}$), then we see that the right-hand side vanishes, so $\bar{\sigma}_2 \cdot \bar{\sigma} = 0$.

Case 3. Finally, suppose $n = 3$. In this case, Equation (2) tells us that $[\bar{\sigma}_3, x(m)] = -mx(m+2)$ for any $x \in \mathfrak{g}$, so it follows that

$$\begin{aligned}
\bar{\sigma}_3 \cdot \bar{\sigma} &= \frac{\kappa_0}{2(\kappa - \kappa_c)}\left([\bar{\sigma}_3, x_i(-1)]x^i(-1) + x_i(-1)[\bar{\sigma}_3, x^i(-1)]\right)|0\rangle \\
&= \frac{\kappa_0}{2(\kappa - \kappa_c)}\left(x_i(1)x^i(-1) + x_i(-1)x^i(1)\right)|0\rangle \\
&= \frac{\kappa_0}{2(\kappa - \kappa_c)}\left(x_i(1)x^i(-1)\right)|0\rangle \\
&= \frac{\kappa_0}{2(\kappa - \kappa_c)}\left([x_i, x^i](-1) + \kappa(x_i, x^i)\right)|0\rangle .
\end{aligned}$$

Once again picking $\{x_i\}$ orthonormal with respect to κ_0, we deduce that

$$\bar{\sigma}_3 \cdot \bar{\sigma} = \frac{\kappa_0}{2(\kappa - \kappa_c)}\kappa(x_i, x^i)\,|0\rangle = \frac{\kappa \cdot \dim \mathfrak{g}}{2(\kappa - \kappa_c)}\,|0\rangle .$$

Hence, our OPE is

$$Y(\sigma, z)Y(\sigma, w) \sim \frac{\partial_w Y(\sigma, w)}{(z-w)} + \frac{2Y(\sigma, w)}{(z-w)^2} + \frac{c_\kappa/2}{(z-w)^4},$$

where $c_\kappa = \frac{\kappa \cdot \dim \mathfrak{g}}{\kappa - \kappa_c}$. By comparing coefficients on both sides (see the conseequences of associativity in the second part of Ilya's notes), the desired commutation relations follow from this OPE. □

It follows that the Sugawara operators span a Lie subalgebra of End $V_\kappa(\mathfrak{g})$ isomorphic to a quotient of the Virasoro algebra Vir with central charge given by c_κ. The fact that the elements $\{L_n\}_{n\in\mathbb{Z}}$ are indeed linearly independent in End $V_\kappa(\mathfrak{g})$ is an exercise.

Let us now make a brief digression on the Virasoro algebra. The Witt algebra has a unique (up to scaling) nontrivial 2-cycle, and the corresponding 1-dimensional central extension is the Virasoro algebra

$$\mathrm{Vir} = W \oplus \mathbb{C}C,$$

with bracket relations such that C is central and

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3 - m}{12}\delta_{m+n,0}C$$

for any $m, n \in \mathbb{Z}$. Similar to the vacuum module in the affine Kac–Moody case, we can construct a vertex algebra associated to the Virasoro algebra. Fix $c \in \mathbb{C}$. Then, let $\mathbb{C}_c$ be the one-dimensional representation of $\mathbb{C}[[t]]\partial_t \oplus \mathbb{C}C \subset$ Vir where $\mathbb{C}[[t]]\partial_t$ acts by zero and C acts by c. Define the Virasoro vacuum module as the induced representation

$$\mathrm{Vir}_c := U(\mathrm{Vir}) \otimes_{\mathbb{C}[[t]]\partial_t \oplus \mathbb{C}C} \mathbb{C}_c.$$

We can give Vir_c a natural vertex algebra structure in a way analogous to the vacuum module $V_\kappa(\mathfrak{g})$. Observe that, by the Poincaré–Birkhoff–Witt theorem, Vir_c has a basis given by

$$L_{i_1}L_{i_2}\cdots L_{i_n}\,|0\rangle ,$$

where $|0\rangle = 1 \otimes 1$ and $i_1 \leq i_2 \leq \cdots \leq i_n < -1$ are integers. Let us define the field

$$L(z) := \sum_{n \in \mathbb{Z}} L_n z^{-n-2}.$$

Note that $L(z)$ satisfies essentially the same OPE as the renormalized Sugawara field $Y(\bar{\sigma}, z) \in \operatorname{End} V_\kappa(\mathfrak{g})((z))$ since the renormalized Sugawara operators L_n satisfy the same mutual commutation relations as the Fourier coefficients L_n. That is,

$$L(z)L(w) \sim \frac{c/2}{(z-w)^4} + \frac{2L(w)}{(z-w)^2} + \frac{\partial_w L(w)}{(z-w)}.$$

Hence, we see that $L(z)$ is mutually local with itself, and after setting $T = -\partial_t$, we can apply the reconstruction theorem (see page 12 in Ilya's first set of notes) to deduce the following result.

Proposition 1.5. [Fre07, Theorem 3.1.1] *There exists a unique vertex algebra structure on* Vir_c *such that the vacuum vector is given by* $|0\rangle$ *and* $Y(L_{-2}|0\rangle, z) = L(z)$. *More precisely, the state-field correspondence is given by*

$$Y(L_{-1-i_1} \cdots L_{-1-i_n}|0\rangle, z) = \frac{1}{(i_1-1)! \cdots (i_n-1)!} : \partial_z^{i_1} L(z) \cdots \partial_z^{i_n} L(z) :$$

for any $i_1 \geq i_2 \geq \cdots \geq i_n \geq 0$.

1.2. **Conformal Vertex Algebras.** We will now introduce a structure on vertex algebras that both the Virasoro vertex algebra and the Kac–Moody vertex algebra possess. Namely, observe that the Kac–Moody vertex algebra (away from the critical level) contains a vector $\bar{\sigma}$ such that the Fourier coefficients of $Y(\bar{\sigma}, z)$ behave like the Fourier coefficients of the field $L(z)$. In other words, these vertex algebras admit an internal action of the Virasoro algebra, a symmetry that recurs in many interesting vertex algebras. The Virasoro algebra acts by infinitesimal generators of the conformal transformations of the Riemann sphere, so these vertex algebras are given the name "conformal." We axiomatize this structure with the following definition.

Definition 1.6. A **conformal vertex algebra** of central charge $c \in \mathbb{C}$ is a $\mathbb{Z}$-graded vertex algebra V containing a degree two vector $\nu \in V$, called a *conformal vector*, such that

(i) the Fourier coefficients L_n^ν in
$$Y(\nu, z) = \sum_{n \in \mathbb{Z}} L_n^\nu z^{-n-2}$$
satisfy the defining relations of the Virasoro algebra with central charge c,

(ii) the translation operator T must coincide with the coefficient L_{-1}^ν,

(iii) the coefficient L_0^ν is the grading operator, that is, it acts by scalar multiplication by n on the degree n graded component of V.

Note that the first condition is equivalent to $Y(\nu, z)$ satisfying the OPE

$$Y(\nu, z)Y(\nu, w) \sim \frac{c/2}{(z-w)^4} + \frac{2Y(\nu, w)}{(z-w)^2} + \frac{\partial_w Y(\nu, w)}{(z-w)}.$$

Observe that every conformal vertex algebra of central charge c can is a representation of Vir with central charge c. The vacuum module $V_\kappa(\mathfrak{g})$ is conformal of central charge

c_κ when κ is non-critical, and the conformal vector is given by $\bar{\sigma}$. Similarly, the Virasoro vacuum module Vir_c is tautologically conformal of central charge c.

Perhaps unsurprisingly, Vir_c plays an important role among conformal vertex algebras of central charge c. More precisely, we will soon show that any conformal vertex algebra of central charge c admits a unique homomorphism from Vir_c. This homomorphism is not only a Vir-module homomorphism but also a *vertex algebra homomorphism*.

Definition 1.7. A **homomorphism of vertex algebras** is a linear map $\varphi : V_1 \to V_2$ with

(i) $\varphi(|0\rangle_1) = |0\rangle_2$,
(ii) $\varphi \circ T_1 = T_2 \circ \varphi$, and
(iii) for any $A, B \in V_1$, we have $\varphi(Y_1(A,z)B) = Y_2(\varphi(A),z)\varphi(B)$.

Before we proceed to the following proposition, we make a few simple but crucial observations regarding any conformal vertex algebra V. From $Y(\nu,z)\,|0\rangle = \nu$, we can deduce that $\nu = L^\nu_{-2}\,|0\rangle$. In fact, applying the strong reconstruction theorem to the subspace spanned by the vectors of the form $L^\nu_{-1-i_1}\cdots L^\nu_{-1-i_n}\,|0\rangle$ (for $i_1 \geq \cdots \geq i_n \geq 0$) with the field $Y(\nu,z)$ allows us to deduce that

$$Y(L^\nu_{-1-i_1}\cdots L^\nu_{-1-i_n}\,|0\rangle\,,z) = \frac{1}{(i_1-1)!\cdots(i_n-1)!} :\partial_z^{i_1}Y(\nu,z)\cdots\partial_z^{i_n}Y(\nu,z): \tag{3}$$

for any $i_1 \geq i_2 \geq \cdots \geq i_n \geq 0$.

Proposition 1.8. [Fre07, Lemma 3.1.2] *If V is a conformal vertex algebra of central charge c with conformal vector ν, then there exists a unique homomorphism of vertex algebras $\mathrm{Vir}_c \to V$ sending $L_{-2}\,|0\rangle$ to ν. This linear map is also a homomorphism of* Vir*-modules.*

Proof. Observe that such a vertex algebra homomorphism must be a homomorphism of Vir-modules. Indeed, given any $A \in \mathrm{Vir}_c$, the relation $\varphi(L_j A) = L^\nu_j\varphi(A)$ (for any $j \in \mathbb{Z}$) immediately follows from comparing coefficients on both sides of

$$\varphi(Y(L_{-2}\,|0\rangle\,,z)A) = Y(\nu,z)\varphi(A).$$

Since $|0\rangle$ generates Vir_c as a Vir-module, it follows that such a vertex algebra homomorphism is uniquely determined after we know the corresponding action of Vir on V. This action is determined by the image of $L_{-2}\,|0\rangle$, so it follows that φ is uniquely determined by the image of $L_{-2}\,|0\rangle$. It remains to prove that such a homomorphism exists. We simply consider the map $\mathrm{Vir}_c \to V$ given by

$$L_{i_1}L_{i_2}\cdots L_{i_n}\,|0\rangle \mapsto L^\nu_{i_1}L^\nu_{i_2}\cdots L^\nu_{i_n}\,|0\rangle\,,$$

which is easily seen to be a vertex algebra homomorphism thanks to (3). □

We conclude this section with a sufficient condition for a vector in a $\mathbb{Z}_{\geq 0}$-graded vertex algebra to be conformal.

Proposition 1.9. [Fre07, Lemma 3.1.2] *A $\mathbb{Z}_{\geq 0}$-graded vertex algebra V is conformal with central charge c if and only if there exists a (nonzero) degree two vector $\nu \in V$ such that the coefficients L^ν_n in $Y(\nu,z) = \sum_{n\in\mathbb{Z}} L^\nu_n z^{-n-2}$ satisfy the following conditions:*

(1) $L^\nu_{-1} = T$,

(2) L_0^ν *acts as the grading operator,*
(3) $L_2^\nu \cdot \nu = (c/2)\,|0\rangle$.

Proof. For the forward direction, conditions (1) and (2) follow by definition. For condition (3), we recall that $\nu = L_{-2}^\nu\,|0\rangle$. It follows that

$$L_2^\nu \nu = [L_2^\nu, L_{-2}^\nu]\,|0\rangle = (L_0^\nu + \frac{c}{2})\,|0\rangle = \frac{c}{2}\,|0\rangle\,,$$

as needed.

Conversely, suppose V contains a nonzero $\nu \in V_2$ such that the Fourier coefficients of $Y(\nu, z)$ satisfy the conditions above. We need to show that the coefficients L_n^ν satisfy the defining relations of the Virasoro algebra. By the OPE formalism from before, it suffices to show that $Y(\nu, z)$ satisfies the following OPE:

$$Y(\nu,z)Y(\nu,w) \sim \frac{c/2}{(z-w)^4} + \frac{2Y(\nu,w)}{(z-w)^2} + \frac{\partial_w Y(\nu,w)}{(z-w)}.$$

By definition,

$$Y(\nu,z)Y(\nu,w) \sim \sum_{n \geq 0} \frac{Y(L_{n-1}^\nu \nu, w)}{(z-w)^{n+1}}.$$

Note that $L_{n-1}^\nu \nu$ has degree $(1-n)+2 = 3-n$. Since the degrees of V are nonnegative, it suffices to consider only the cases $0 \leq n \leq 3$. In the case $n = 0$, we have $L_{-1}^\nu \nu = T\nu$, it follows that $Y(L_{-1}^\nu \nu, w) = \partial_w Y(\nu, w)$. Similarly, we are given $L_0^\nu \nu = \deg \nu \cdot \nu = 2\nu$, and $L_2^\nu \nu = (c/2)\,|0\rangle$. Hence, it follows that

$$Y(\nu,z)Y(\nu,w) \sim \frac{c/2}{(z-w)^4} + \frac{\alpha(w)}{(z-w)^3} + \frac{2Y(\nu,w)}{(z-w)^2} + \frac{\partial_w Y(\nu,w)}{(z-w)}, \tag{4}$$

where $\alpha(w) := Y(L_1\nu, w)$. It remains to show that $\alpha(w) = 0$. By swapping the roles of z and w, we obtain

$$Y(\nu,w)Y(\nu,z) \sim \frac{c/2}{(z-w)^4} - \frac{\alpha(z)}{(w-z)^3} + \frac{2Y(\nu,z)}{(z-w)^2} - \frac{\partial_w Y(\nu,w)}{(z-w)}. \tag{5}$$

By locality, the right-hand sides of (4) and (5) must be equal to each other. If we perform a Taylor series expansion on the right-hand side of (5) in terms of w and take only the singular terms (at $z = w$), we have

$$\frac{c/2}{(z-w)^4} - \frac{\alpha(w) + \partial_w \alpha(w)(z-w) + \partial_w^2 \alpha(w)(z-w)^2}{(z-w)^3} + O((z-w)^{-2}).$$

In particular, the coefficient of $(z-w)^{-3}$ when the right-hand side of (5) is expanded in terms of w is $-\alpha(w)$. It follows that $\alpha(w) = 0$, completing the proof. □

2. Lie Algebra of Fourier Coefficients

This section forms the centerpiece of the talk (both literally and spiritually). Our work in this section will demonstrate formally recover the completed universal enveloping algebra from the affine vertex algebra, which will be crucial for more complicated computations in the future. To do so, we will introduce a Lie algebra $\widetilde{F}_V$ consisting of "formal" symbols representing the Fourier coefficients of the vertex operators on any vertex algebra. We will then produce a suitable homomorphism from $\widetilde{F}_V$ to the completed universal enveloping algebra. We conclude this section by studying the universal enveloping algebra of $\widetilde{F}_V$. By completing this universal enveloping algebra and quotienting out by some relations, we will show that $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$ can be completely recovered from the vertex algebra structure encoded in $\widetilde{F}_V$.

As a byproduct of this framework, we will be able to translate our commutation relations between the Sugawara operators into commutation relations between the Sugawara elements in $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$. In particular, we will be able to show that the Sugawara elements do indeed belong to the center $Z_{\kappa_c}(\hat{\mathfrak{g}})$ at the critical level.

2.1. **Lie Algebra of Formal Fourier Coefficients.**

Consider the action homomorphism

$$\widetilde{U}_\kappa(\hat{\mathfrak{g}}) \to \operatorname{End} V_\kappa(\mathfrak{g}), \tag{6}$$

which sends each $x_i(n)$ to the endomorphism $x_i(n)$. If the homomorphism (6) were injective, then we would immediately deduce that the relations for endomorphisms also hold for elements of $\widetilde{U}_\kappa(\mathfrak{g})$. Unfortunately, this homomorphism is not injective at the critical level: each Sugawara operator S_N for $N \geq -1$ annihilates the vacuum vector $|0\rangle$, and since the Sugawara operators are central, it follows that they act by zero on the entire vacuum module. It seems to be the case that this homomorphism is injective for non-critical levels κ, but unfortunately, I do not know about the extent to which this statement is true.

Instead, our strategy will be to consider the Lie (sub)algebra formed by formal symbols representing the Fourier coefficients of vertex operators on $V_\kappa(\mathfrak{g})$. We recovered our commutation relations in the previous section by implicitly studying this Lie algebra, i.e., by doing formal computations using the OPE formalism. We'll interpret our present commutation relations as relations in an abstract Lie algebra and show that these relations also hold in $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$.

Definition 2.1. For an arbitrary vertex algebra V, define the vector space

$$F_V := (V \otimes \mathbb{C}[t, t^{-1}]) / \operatorname{im} \partial,$$

where ∂ is the map $T \otimes 1 + 1 \otimes \partial_t$.

We write $A_{[n]} := A \otimes t^n$ for each $A \in v$ and $n \in \mathbb{Z}$, so that F_V is spanned by the collection of all $A_{[n]}$. We want the formal symbol $A_{[n]}$ to emulate the Fourier coefficient A_n in the vertex operator

$$Y(A, z) = \sum_{n \in \mathbb{Z}} A_n z^{-n-1}.$$

More precisely, there is a natural map $V \otimes \mathbb{C}[t, t^{-1}] \to \operatorname{End} V$ given by $A \otimes t^n \mapsto A_n$. Since the nth Fourier coefficient of $[T, Y(A, z)] = \partial_z Y(A, z)$ is precisely $-nA_{n-1}$, this map vanishes on $\operatorname{Im} \partial$ and gives us a linear map $F_V \to \operatorname{End} V$ via $A_{[n]} \mapsto A_n$.

Moreover, recall that the Fourier coefficients of vertex operators on V form a Lie algebra, with Lie bracket (1):

$$[A_m, B_n] = \sum_{\ell \geq 0} \binom{m}{\ell} (A_\ell B)_{m+n-\ell}$$

for all $A, B \in V$ and $m, n \in \mathbb{Z}$. Motivated by this formula, we shall define a bilinear map $[-,-] : F_V \times F_V \to F_V$ via

$$[A_{[m]}, B_{[n]}] := \sum_{\ell \geq 0} \binom{m}{\ell} (A_\ell B)_{[m+k-\ell]}. \tag{7}$$

Note that the sum on the right-hand side is actually a finite sum since $A_n B = 0$ for sufficiently large n.

In the following proposition, we show that this bilinear map is a Lie bracket on F_V. To do so, we will need the notion of a tensor product of vertex algebras.

Definition 2.2 (and Lemma)**.** Suppose V_1 and V_2 are vertex algebras. The tensor product $V_1 \otimes_{\mathbb{C}} V_2$ admits a unique vertex algebra structure with the data

(i) $|0\rangle := |0\rangle_1 \otimes |0\rangle_2$,
(ii) $T := T_1 \otimes \mathrm{Id} + \mathrm{Id} \otimes T_2$,
(iii) for any $A_1 \in V_1$ and $A_2 \in V_2$,
$$Y(A_1 \otimes A_2, z) := Y(A_1, z) \otimes Y(A_2, z) = \sum_{n \in \mathbb{Z}} \sum_{m+\ell=n} [(A_1)_m \otimes (A_2)_\ell] z^{-n-2}.$$

Proof. Left as an exercise for the reader. Note that the infinite sum of operators

$$\sum_{m+\ell=n} (A_1)_m \otimes (A_2)_\ell$$

for fixed $n \in \mathbb{Z}$ is a well-defined operator on $V_1 \otimes V_2$: indeed, given $v_1 \otimes v_2 \in V_1 \otimes V_2$, we have $(A_1)_m v_1 = 0$ for $m \gg 0$ and $(A_2)_\ell v_2 = 0$ for $\ell \gg 0$, so the sum

$$\sum_{m+\ell=n} (A_1)_m v_1 \otimes (A_2)_\ell v_2$$

contains only finitely many nonzero terms. □

Let's proceed to the proposition regarding F_V.

Lemma 2.3. [Fre07, Proposition 3.2.1 (Chapter 3.2.1)] *The bilinear map defined by (7) is a Lie bracket on F_V. Moreover, the natural linear map $F_V \to \mathrm{End}\, V_\kappa(\mathfrak{g})$ given by $A_{[n]} \mapsto A_n$ is a Lie algebra homomorphism.*

Proof. For an arbitrary vertex algebra W, let $F_W[0]$ be the subspace of F_W spanned by the symbols $A_{[0]}$ for $A \in W$. Note that $F_W[0]$ is closed under $[-,-]$, as

$$[A_{[0]}, B_{[0]}] = (A_0 B)_{[0]}.$$

Let us show that the restriction of $[-,-]$ to a map $F_W[0] \times F_W[0] \to F_W[0]$ defines a Lie bracket on $F_W[0]$. From skew symmetry (see page 3 in Ilya's second set of notes),

$$Y(A, z)B = e^{zT} Y(B, -z)A.$$

Comparing the coefficients of z^{-1} on both sides, we have

$$A_0B = \left(\sum_{n\geq 0} \frac{(-1)^{n+1}T^n B_n}{n!} \right) A = -B_0A + TB_1A - T^2B_2A/2 + \cdots$$

Recall that $B_nA = 0$ for $n \gg 0$. Hence, there is a vector $w \in W$ such that

$$A_0B = -B_0A + Tw. \tag{8}$$

Tensoring with 1 on both sides and considering the image in F_W gives

$$[A_{[0]}, B_{[0]}] = (A_0B)_{[0]} = -(B_0A)_{[0]} + Tw \otimes 1 = -(B_0A)_{[0]} = -[B_{[0]}, A_{[0]}],$$

which shows that $[-,-]$ is antisymmetric when restricted to $F_W[0]$.

It remains to prove the Jacobi identity, or equivalently,

$$((A_0B)_0C)_{[0]} + ((C_0A)_0B)_{[0]} + ((B_0C)_0A)_{[0]} = 0.$$

From equation (1), we have

$$(A_0B)_0 = [A_0, B_0], \quad (C_0A)_0 = [C_0, A_0], \quad (B_0C)_0 = [B_0, C_0].$$

By Equation (8), there exists some $w \in W$ such that

$$[A_0, B_0]C - B_0(C_0A) - A_0(B_0C) = A_0B_0C - B_0A_0C + B_0A_0C - A_0B_0C + Tw = Tw.$$

Tensoring with 1 on both sides and taking the image on F_W gives

$$\begin{aligned} 0 &= ([A_0, B_0]C)_{[0]} - (B_0(C_0A))_{[0]} - (A_0(B_0C))_{[0]} \\ &= ((A_0B)_0C)_{[0]} + ((C_0A)_0B)_{[0]} + ((B_0C)_0A)_{[0]}, \end{aligned}$$

which is precisely the Jacobi identity.

Now, let us find a vertex algebra W such that $F_V = F_W[0]$ and so that the restriction of the map $[-,-]_W$ on F_W to $F_W[0]$ coincides with the map $[-,-]_V$ on F_V. Note that the commutative associative algebra $\mathbb{C}[w, w^{-1}]$ equipped with the derivation $-\partial_w$ has the natural structure of a vertex algebra. Hence, the tensor product $W := V \otimes_{\mathbb{C}} \mathbb{C}[w, w^{-1}]$ also admits a natural vertex algebra structure mentioned in Definition 2.2. We have a natural isomorphism $F_V \xrightarrow{\sim} F_W[0]$ given by $A_{[n]} \mapsto (A \otimes w^n)_{[0]}$.

To complete the proof, we need to show that

$$[A_{[m]}, B_{[n]}] \mapsto [(A \otimes w^m)_{[0]}, (B \otimes w^n)_{[0]}]_W = ((A \otimes w^m)_0(B \otimes w^n))_{[0]}.$$

The image of $[A_{[m]}, B_{[n]}]$ is given by

$$\sum_{\ell \geq 0} \binom{m}{\ell} (A_\ell B \otimes w^{m+n-\ell})_{[0]}.$$

By definition, the coefficient of z^{-1} in $Y(A \otimes w^m, z) = Y(A, z) \otimes (z + w)^m$ is given by

$$(A \otimes w^m)_0 = \sum_{\ell \geq 0} \binom{m}{\ell} A_n w^{m-\ell},$$

so it follows that

$$(A \otimes w^m)_0(B \otimes w^n) = \sum_{\ell \geq 0} \binom{m}{\ell} (A_\ell B \otimes w^{m+n-\ell})_{[0]},$$

as desired. The claim on the Lie algebra homomorphism $F_V \to \operatorname{End} V_\kappa(\mathfrak{g})$ follows immediately from comparing Equations (7) and (1). □

In fact, suitable infinite sums of Fourier coefficients are well-defined elements of $\operatorname{End} V$, so we will emulate this behavior by considering a completion of F_V, denoted $\widetilde{F}_V$. In the Kac–Moody case, we will see that this formalism allow us to describe any element of $\widetilde{F}_V$ in terms of infinite sums of products of the basis elements $x_i(n)$.

Definition 2.4. Let $\widetilde{F}_V := V \otimes \mathbb{C}((t))/\operatorname{im}\partial$ be the completion of F_V with respect to the natural topology on $\mathbb{C}[t, t^{-1}]$, i.e., so $\widetilde{F}_V$ is spanned by infinite sums of the form $\sum_{n\geq N} c_n A_{[n]}$ for $c_n \in \mathbb{C}$ and $N \in \mathbb{Z}$.

A straightforward exercise shows that the Lie bracket on F_V extends via continuity to $\widetilde{F}_V$. Thus, the map $[-,-]$ endows $\widetilde{F}_V$ with the structure of a Lie algebra by continuity.

Moreover, for every $v \in V$, we have $A_n v = 0$ for $n \gg 0$, so it follows that the homomorphism $F_V \to \operatorname{End} V_\kappa(\mathfrak{g})$ extends by continuity to a homomorphism $\widetilde{F}_V \to \operatorname{End} V_\kappa(\mathfrak{g})$. This homomorphism can be expressed in simple terms: any element of the form $\sum_{n\geq N} c_n A_{[n]} \in \widetilde{F}_V$ is mapped to

$$\sum_{n\geq N} c_n A_n = \operatorname{Res}_{z=0} Y(A,z) \cdot \sum_{n\geq N} c_n z^n \, dz.$$

We conclude this part with some straightforward remarks.

Remark 2.5. If $\phi : V_1 \to V_2$ is a homomorphism of vertex algebras, then note that we have a natural map of (completed) Lie algebras $\widetilde{F}_{V_1} \to \widetilde{F}_{V_2}$ given by $A_{[0]} \mapsto \phi(A)_{[0]}$. Moreover, this construction is natural, so $V \mapsto \widetilde{F}_V$ defines a covariant functor from the category of vertex algebras to the category of complete topological Lie algebras.

Remark 2.6. Finally, if V is a $\mathbb{Z}$-graded vertex algebra, then we can give $\widetilde{F}_V$ the structure of a $\mathbb{Z}$-graded Lie algebra by assigning $\deg A_{[n]} = -n + \deg A + 1$ for all $A \in V$ (emulating the degrees of Fourier coefficients of the corresponding vertex operator).

Remark 2.7. Recall the vector

$$\sigma = \frac{1}{2} \sum_{i=1}^{\dim \mathfrak{g}} x_i(-1) x^i(-1) |0\rangle \in V_\kappa(\mathfrak{g}).$$

Since our calculations in Section 1 relied only on Equation (1), they formally carry over to $\widetilde{F}_V$ via Equation (7). That is,

$$[\sigma_{[n]},\ (x_i(-1)\,|0\rangle)_{[m]}] = m\left(\frac{\kappa_c - \kappa}{\kappa_0}\right)(x_i(-1)\,|0\rangle)_{[n+m-1]} \tag{9}$$

and moreover, if we let $\bar{\sigma} := \frac{\kappa_0}{\kappa - \kappa_c}\sigma$ and $c_\kappa = \dim \mathfrak{g} \cdot \frac{\kappa}{\kappa - \kappa_c}$ when $\kappa \neq \kappa_c$ is not critical,

$$[\bar{\sigma}_{[n+1]}, \bar{\sigma}_{[m+1]}] = (n-m)\bar{\sigma}_{[n+m+1]} + \frac{n^3 - n}{12} c_\kappa \delta_{n+m,0}. \tag{10}$$

2.2. **Homomorphism into the Completed Universal Enveloping Algebra.** We now demonstrate how the formalism of the previous section can be used to translate our results from the vertex algebra to the universal enveloping algebra. Our goal will be to produce a Lie algebra homomorphism $\widetilde{F}_{V_\kappa(\mathfrak{g})} \to \widetilde{U}_\kappa(\hat{\mathfrak{g}})$ under which the elements $(x_i(-1)\,|0\rangle)_{[m]}$ are mapped to $x_i(m)$. Recalling our motivation for defining $\widetilde{F}_{V_\kappa(\mathfrak{g})}$, we will produce such a map by informally considering the elements of $\widetilde{F}_V$ as Fourier coefficients of vertex operators.

Recall that $V_\kappa(\mathfrak{g})$ has a basis given by elements of the form

$$A_{\mathbf{i}}(\mathbf{n}) := x_{i_1}(n_1)\cdots x_{i_k}(n_k)\,|0\rangle \in V_\kappa(\mathfrak{g}). \tag{11}$$

The state-field correspondence assigns this element to the normally ordered product

$$Y(A_{\mathbf{i}}(\mathbf{n}), z) = \prod_{j=1}^{k} \frac{1}{(-n_j-1)!} : \partial_z^{-n_1-1} X_{i_1}(z) \cdots \partial_z^{-n_k-1} X_{i_k}(z) : . \tag{12}$$

The Fourier coefficients in this field are given by

$$A_{\mathbf{i}}(\mathbf{n})_m = \sum_{a_1+\cdots+a_k=m-k+1} \left(\prod_{\ell=1}^{k} \binom{a_\ell - n_\ell - 1}{-n_\ell - 1} \right) \left\{ \prod_{\ell=1}^{k} x_\ell(a_\ell - n_\ell - 1) \right\}, \tag{13}$$

where the notation $\{-\}$ indicates that the terms in the product are possibly ordered according to the normal ordering in (12). Arguing by induction on k, one can show that the right-hand side of (13) can be viewed as a bona fide element of the completed universal enveloping algebra $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$. Essentially, the normal ordering ensures that all but finitely many terms have $x_i(n)$ with $n > N$ on the right for every $N \in \mathbb{Z}$, giving rise to an infinite sum belonging to $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$.

For any $m \in \mathbb{Z}$, we assign the element $A_{\mathbf{i}}(\mathbf{n})_{[m]} \in F_V$ to the right-hand side of Equation (13), now regarded as an element of $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$. Since the elements of the form $A_{\mathbf{i}}(\mathbf{n})_{[m]}$ form a basis for F_V, we can extend this assignment to a linear map $F_{V_\kappa(\hat{\mathfrak{g}})} \to \widetilde{U}_\kappa(\hat{\mathfrak{g}})$. Subsequently extending via continuity, we have a linear map $\varphi : \widetilde{F}_{V_\kappa(\mathfrak{g})} \to \widetilde{U}_\kappa(\hat{\mathfrak{g}})$.

Example 2.8. Fix some $i = 1, \ldots, \dim \mathfrak{g}$. The mth Fourier coefficient in the vertex operator $Y(x_i(-1)\,|0\rangle\,, z)$ is the operator $x_i(m)$, which can clearly be regarded as an element of $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$. Therefore, our map φ sends $(x_i(-1)\,|0\rangle)_{[m]} \in \widetilde{F}_V$ to the element $x_i(m) \in \widetilde{U}_\kappa(\hat{\mathfrak{g}})$.

Example 2.9. Recall the element

$$\sigma = \frac{1}{2} \sum_{i=1}^{\dim \mathfrak{g}} x_i(-1) x^i(-1)\,|0\rangle \in V_\kappa(\mathfrak{g}).$$

For each fixed $i = 1, \ldots, \dim \mathfrak{g}$ and $n \in \mathbb{Z}$, the $(n-1)$st Fourier coefficient of the vertex operator $Y(x_i(-1)x^i(-1)\,|0\rangle\,, z)$ is given by

$$(x_i(-1)x^i(-1)\,|0\rangle)_{n-1} = \sum_{m+\ell=n} : x_i(m) x^i(\ell) :$$

Observe that the right-hand side can be regarded as an element of $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$. Hence, by linearity, our map φ sends $\sigma_{[n-1]}$ to

$$\frac{1}{2}\sum_{i=1}^{\dim \mathfrak{g}} \sum_{m+\ell=n} :x_i(m)x^i(\ell): \,=S_n \in \widetilde{U}_\kappa(\hat{\mathfrak{g}}),$$

the nth Sugawara element.

Remark 2.10. From its definition, observe that the map φ has the following property: the composition

$$\widetilde{F}_{V_\kappa(\mathfrak{g})} \xrightarrow{\varphi} \widetilde{U}_\kappa(\hat{\mathfrak{g}}) \to \operatorname{End} V_\kappa(\mathfrak{g}),$$

where the second map is the action homomorphism, is precisely the Lie algebra homomorphism $\widetilde{F}_{V_\kappa(\mathfrak{g})} \to \operatorname{End} V_\kappa(\mathfrak{g})$ given by $A_{[n]} \mapsto A_n$.

We now state and prove the main result regarding the map φ.

Proposition 2.11. [FBZ04, Proposition 4.2.2] *The natural linear map $\varphi : \widetilde{F}_{V_\kappa(\mathfrak{g})} \to \widetilde{U}_\kappa(\hat{\mathfrak{g}})$ is a Lie algebra homomorphism.*

The proof of this proposition is rather tedious, so we break it into digestible chunks. Unsurprisingly, we will be particularly interested in formal power series with coefficients in $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$, so we begin with some auxiliary results regarding these series.

Lemma 2.12 (Lemma 1.1.4, [FBZ04]). *Take $f(z,w) \in \widetilde{U}_\kappa(\hat{\mathfrak{g}})[[z^{\pm 1}, w^{\pm 1}]]$ and suppose there exists $N \geq 0$ such that $(z-w)^N f(z,w) = 0$. Then, we can write $f(z,w)$ uniquely as a sum*

$$\sum_{i=0}^{N-1} g_i(w)\partial_w^i \delta(z-w),$$

for some formal series $g_i(w) \in \widetilde{U}_\kappa(\hat{\mathfrak{g}})[[w^{\pm 1}]]$.

Proof. Use formal properties of the delta function and its derivatives. See Exercise 4 in Ilya's second set of notes. Although the exercise was stated when the base ring is End V instead of $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$, it applies in this situation as well. □

Corollary 2.13. *Suppose $f(z), h(z) \in \widetilde{U}_\kappa(\hat{\mathfrak{g}})[[z^{\pm 1}]]$ satisfy $(z-w)^N[f(z), h(w)] = 0$ for some $N \geq 0$. Then, we can write*

$$[f(z), h(w)] = \sum_{i=0}^{N-1} \frac{1}{i!} g_i(w)\partial_w^i \delta(z-w),$$

where

$$g_i(w) = \operatorname{Res}_{z=0}((z-w)^i [f(z), h(w)]).$$

Proof. This corollary follows from the identity

$$\operatorname{Res}_{z=0} \frac{1}{i!}(z-w)^j \partial_w^i \delta(z-w) = \delta_{i,j}$$

for any $i, j \geq 0$ as well as Lemma 2.12. □

Definition 2.14. For any $A \in V_\kappa(\mathfrak{g})$, define the formal series

$$Y'(A,z) := \sum_{n\in\mathbb{Z}} \varphi(A_{[n]})z^{-n-1} \in \widetilde{U}_\kappa(\hat{\mathfrak{g}})[[z^{\pm 1}]].$$

We will show that Corollary 2.13 applies to these formal series. Although we defined normally ordered products in the setting of fields, note that the definition generalizes to formal series in $\widetilde{U}_\kappa(\hat{\mathfrak{g}})[[z^{\pm 1}]]$. That is, given $f(z), g(w) \in \widetilde{U}_\kappa(\hat{\mathfrak{g}})[[z^{\pm 1}]]$, we can define

$$: f(w)g(w) : \ = \mathrm{Res}_{z=0}\left(\left(\frac{1}{z-w}\right)_{|z|>|w|} f(z)g(w) - \left(\frac{1}{z-w}\right)_{|w|>|z|} g(w)f(z)\right)$$

Lemma 2.15. *For any $A, B \in V_\kappa(\mathfrak{g})$, there exists $N > 0$ such that*

$$(z-w)^N[Y'(A,z), Y'(B,w)] = 0.$$

In this case, we say that $Y'(A,z)$ and $Y'(B,z)$ are local.

Proof. Observe that the claim is true when $A = x_i(-n)\,|0\rangle$ and $B = x_j(-m)\,|0\rangle$ for some $i, j = 1, \ldots, \dim \mathfrak{g}$ and $n, m \in \mathbb{Z}_{\geq 0}$. In this case, we have

$$Y'(A,z) = \frac{1}{(n-1)!}\partial_z^{n-1}\widetilde{X}_i(z), \quad Y'(B,z) = \frac{1}{(m-1)!}\partial_z^{m-1}\widetilde{X}_j(z),$$

where

$$\widetilde{X}_k(z) := \sum_{n\in\mathbb{Z}} x_i(n)z^{-n-1} \in \widetilde{U}_\kappa(\hat{\mathfrak{g}})[[z^{\pm 1}]].$$

Thanks to the commutation relations in $\hat{\mathfrak{g}}$, we have the following identity:

$$[\widetilde{X}_i(z), \widetilde{X}_j(w)] = \widetilde{[x_i, x_j]}(w)\delta(z-w) + \kappa(x_i, x_j)\partial_w\delta(z-w).$$

In particular, $\widetilde{X}_i(z)$ and $\widetilde{X}_j(w)$ are local, so their derivatives $Y'(A,z)$ and $Y'(B,z)$ are also local.

Now, from the definition of φ, observe that any series $Y'(A,z)$ and $Y'(B,z)$ can be expressed as a linear combination of normally ordered products of series of the form above. Then, the result follows from the following generalization of Dong's Lemma:

Claim. If $f(z)$ and $g(z)$ are elements of $\widetilde{U}_\kappa(\hat{\mathfrak{g}})[[z^{\pm 1}]]$ satisfying

$$(z-w)^N[f(z), g(w)] = 0$$

for some $N > 0$, then there exists some $M > 0$ such that

$$(z-w)^M[f(z), : f(w)g(w) :] = (z-w)^M[g(z), : f(w), g(w) :] = 0.$$

Note that the proof of Dong's Lemma immediately generalizes to this setting as well, so the claim above immediately follows. □

Inspired by Corollary 2.13, we write

$$a(w)_{[i]}b(w) := \mathrm{Res}_{z=0}((z-w)^i[a(z), b(w)]) \tag{14}$$

for any power series $a(w), b(w) \in \widetilde{U}_\kappa(\hat{\mathfrak{g}})[[w^{\pm 1}]]$ and any $i \geq 0$.

Lemma 2.16. *For mutually local power series $a(z), b(z), c(z) \in \widetilde{U}_\kappa(\hat{\mathfrak{g}})[[z^{\pm 1}]]$, we have*

$$a(w)_{[m]} : b(w)c(w) : \; = \sum_{j=0}^{m} \binom{m}{j} (a(w)_{[j]} b(w))_{[m-1-j]} c(w) + : b(w)(a(w)_{[m]} c(w)) : .$$

Proof. By Corollary 2.13, we can write

$$[a(z), b(w)] = \sum_{i \geq 0} \frac{1}{i!} (a(w)_{[i]} b(w)) \partial_w^i \delta(z-w).$$

Moreover, recall that

$$: b(w)c(w) : \; = \operatorname{Res}_{y=0} \left(\left(\frac{1}{x-w} \right)_{|x|>|w|} b(z)c(w) - \left(\frac{1}{x-w} \right)_{|w|>|x|} c(w)b(z) \right).$$

The result will follow after expanding

$$a(w)_{[m]} : b(w)c(w) : \; = \operatorname{Res}_{z=0} (z-w)^m [a(z), : b(w)c(w) :]$$

using these identities. The remaining computation is straightforward but tedious, so we refer the reader to [FBZ04, Lemma 4.2.4] for the details. □

With these preliminary facts in mind, we proceed to the proof of Proposition 2.11.

Proof of Proposition 2.11. We wish to show that

$$[\varphi(A_{[m]}), \varphi(B_{[n]})] = \sum_{\ell \geq 0} \binom{m}{\ell} \varphi((A_\ell B)_{[m+n-\ell]})$$

for any $A, B \in V_\kappa(\mathfrak{g})$ and $n, m \in \mathbb{Z}$. Consider the formal series $Y'(A,z)$ and $Y'(B,z)$ defined earlier. For concision, we write

$$G_i^{A,B}(w) := Y'(A,w)_{[i]} Y'(B,w) = \operatorname{Res}_{z=0}((z-w)^i [Y(A,z), Y(B,w)])$$

for any $i \geq 0$. Then, by Lemma 2.15 and Corollary 2.13, we can write

$$[Y'(A,z), Y'(B,w)] = \sum_{i \geq 0} \frac{1}{i!} G_i^{A,B}(w) \partial_w^i \delta(z-w).$$

Comparing coefficients on both sides (similar to Ilya's derivation of Equation (1)),

$$[\varphi(A_{[m]}), \varphi(B_{[n]})] = \sum_{\ell \geq 0} \binom{m}{\ell} (G_\ell^{A,B})_{(m+n-\ell)},$$

where $(G_\ell^{A,B})_{(s)}$ denotes the coefficient of w^{-s-1} in $G_\ell^{A,B}(w)$. Thus, it suffices to prove

$$G_m^{A,B}(w) = Y'(A_m B, w)$$

for any $A, B \in V_\kappa(\mathfrak{g})$ and $m \geq 0$.

We can assume without loss of generality that A and B have the form

$$A = x_{-r_1} \cdots x_{i_t}(-r_t) |0\rangle, \quad B = x_{j_1}(-\ell_1) \cdots x_{j_s}(-\ell_s) |0\rangle.$$

This claim can be proven using induction on $t+s$. First consider the base case where $t = s = 1$. That is, assume $A = x_i(-r)\,|0\rangle$ and $B = x_j(-\ell)\,|0\rangle$. In this case, we have

$$Y'(A,z) = \frac{1}{(r-1)!}\partial_z^{r-1}\widetilde{X}_i(z), \quad Y'(B,z) = \frac{1}{(\ell-1)!}\partial_z^{\ell-1}\widetilde{X}_j(z),$$

where we recall the notation

$$\widetilde{X}_k(z) := \sum_{n\in\mathbb{Z}} x_i(n)z^{-n-1} \in \widetilde{U}_\kappa(\hat{\mathfrak{g}})[[z^{\pm 1}]].$$

Then, the base case follows by differentiating both sides of the identity

$$[\widetilde{X}_i(z), \widetilde{X}_j(w)] = \widetilde{[x_i, x_j]}(w)\delta(z-w) + \kappa(x_i, x_j)\partial_w\delta(z-w),$$

multiplying by $(z-w)^m$ on both sides, taking the residue at $z=0$, and finally comparing coefficients with $Y'(A_mB, w)$.

Now suppose $t \geq 1$ and $s > 1$ are arbitrary. Then, we can write $B = x_j(-\ell)C$ for some vector $C = x_{j_1}(-\ell_1)\cdots x_{j_s}(-\ell_{s-1})\,|0\rangle$. Tracing through the definition of φ, we find that

$$Y'(x_j(-\ell)C, z) = \frac{1}{(\ell-1)!} :\partial_z^{\ell-1}\widetilde{X}_j(z)Y'(C,z): .$$

Then, applying Lemma 2.16 with $a(z) = Y'(A,z)$, $b(z) = \partial_z^{\ell-1}\widetilde{X}_j(z)$, and $c(z) = Y'(C,z)$ along with the induction hypothesis completes the proof. Once again, the calculations are slightly tedious, so we refer the reader to [FBZ04, pg. 68] for the details. □

By passing Equations (9) and (10) through the Lie algebra map $\widetilde{F}_{V_\kappa(\mathfrak{g})} \to \widetilde{U}_\kappa(\hat{\mathfrak{g}})$, we have the following corollary. In short, we can interpret relations in End $V_\kappa(\omega)$ as relations in the completed universal enveloping algebra.

Corollary 2.17. [Fre07, Corollary 3.2.2] *The Sugawara elements S_n satisfy the following commutation relation in $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$:*

$$[S_n, x_i(m)] = \frac{\kappa_c - \kappa}{\kappa_0}\cdot m x_i(n+m),$$

and moreover, when $\kappa \neq \kappa_c$ is not critical,

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{n^3-n}{12}c_\kappa\delta_{n+m,0},$$

where $L_n := \frac{\kappa_0}{\kappa-\kappa_c}\cdot S_n$ and $c_\kappa = \dim\mathfrak{g}\cdot\frac{\kappa}{\kappa-\kappa_c}$. In particular, the Sugawara elements belong to the center $Z_{\kappa_c}(\hat{\mathfrak{g}})$ at the critical level, and when the level κ is not critical, the Sugawara elements generate a Lie subalgebra of $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$ isomorphic to the quotient $\mathrm{Vir}/(C - c_\kappa)$.

For completeness, let us also record the following result for future use. Compare the following result to the relation

$$:Y(A,z)Y(B,z): = Y(A_{-1}B, z)$$

for vertex operators on $V_\kappa(\mathfrak{g})$. The following lemma will give us additional relations will need to impose on the universal enveloping algebra of $\widetilde{F}_{V_\kappa(\mathfrak{g})}$ to establish an isomorphism with $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$.

Lemma 2.18. *For any $A, B \in V_\kappa(\mathfrak{g})$, we have*

$$: Y'(A,z)Y'(B,z) : = Y'(A_{-1}B, z).$$

Proof. Left as a (not-so-fun and slightly tricky) exercise in formal power series. The idea is to use the residue definition of the normally ordered product. Morally, we have

$$: a(w)b(w) : \ \text{“} = \text{''} \ a(w)_{[-1]}b(w),$$

so we essentially want to prove the identity $G_m^{A,B} = Y'(A_m B, z)$ when $m = -1$. Thus, one strategy to the proof is to parallel the proof for $m \geq 0$ from Proposition 2.11. That is, one may want to establish an analog of Lemma 2.16 for nested normally ordered products. Then, assume A and B are monomials and induct on their length. □

2.3. **Recovering the Completed Universal Enveloping Algebra.** Motivated by our result in the previous subsection, we may ask whether we can completely recover the structure of the completed universal enveloping algebra from the data of the vertex algebra given by $\widetilde{F}_V$. In fact, we will now find a positive answer to this question in this section, paving the way for a general framework to study $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$ through the simpler vertex algebra structure on $V_\kappa(\mathfrak{g})$.

More formally, we have defined the Lie algebra of Fourier coefficients $\widetilde{F}_V$, and in the case $V = V_\kappa(\mathfrak{g})$, we found a natural Lie algebra homomorphism $\widetilde{F}_V \to \widetilde{U}_\kappa(\hat{\mathfrak{g}})$. We will soon be able to identify $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$ can be identified with a suitable completion of $U(\widetilde{F}_V)$ modulo some simple relations.

Definition 2.19. For any vertex algebra V, define the completion

$$\widetilde{U}(\widetilde{F}_V) := \varprojlim_N U(\widetilde{F}_V)/I_N,$$

where I_N is the left ideal generated by the elements $A_{[n]}$ for $A \in V$ homogeneous and $n \geq N + \deg A$.

Definition 2.20. Subsequently define $\widetilde{U}(V)$ as the quotient of $\widetilde{U}(\widetilde{F}_V)$ by the relations

$$(A_{-1}B)_{[k]} = \sum_{n+m=k-1} : A_{[n]}B_{[m]} :$$

for all $A, B \in V$ and $k \in \mathbb{Z}$.

In other words, we require an equality of “formal” vertex operators

$$Y[A_{-1}B, z] \ = : Y[A,z]Y[B,z] : \tag{15}$$

where $Y[C,z] = \sum_{n\in\mathbb{Z}} C_{[n]}z^{-n-1}$. The motivation for this equality is the fact that

$$Y'(A_{-1}B, z) = \ : Y'(A,z)Y'(B,z) :$$

in $\widetilde{U}_\kappa(\hat{\mathfrak{g}})[[z^{\pm 1}]]$ by Lemma 2.18.

Let us now specialize to the case $V = V_{\kappa(\mathfrak{g})}$. In this case, we will show that the relations given by Equation (15) are all that are needed to relate $V_\kappa(\mathfrak{g})$ and $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$. In this sense, the completed enveloping algebra $\widetilde{U}(V)$ for an arbitrary vertex algebra V

generalizes the notion of a completed enveloping algebra for the Kac–Moody algebras. Moreover, and of particular importance to our study, this proposition allows us to formally recover the completed universal enveloping algebra $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$ purely from the vertex algebra structure on $V_\kappa(\mathfrak{g})$.

Proposition 2.21. [Fre07, Lemma 3.2.2] *There exists a natural isomorphism of associative algebras $\widetilde{U}(V_\kappa(\mathfrak{g})) \cong \widetilde{U}_\kappa(\hat{\mathfrak{g}})$.*

Proof. Recall that we produced a homomorphism $\varphi : \widetilde{F}_V \to \widetilde{U}_\kappa(\hat{\mathfrak{g}})$. This homomorphism naturally induces a homomorphism of enveloping algebras $U(\widetilde{F}_V) \to \widetilde{U}_\kappa(\hat{\mathfrak{g}})$. Note that this homomorphism in fact extends by continuity to

$$\widetilde{U}(\widetilde{F}_V) \to \widetilde{U}_\kappa(\hat{\mathfrak{g}}).$$

By Lemma 2.18, we see that this map factors through the quotient $\widetilde{U}(\widetilde{F}_V) \to \widetilde{U}(V_\kappa(\mathfrak{g}))$ and induces a homomorphism

$$\widetilde{\varphi} : \widetilde{U}(V_\kappa(\mathfrak{g})) \to \widetilde{U}_\kappa(\mathfrak{g}).$$

Conversely, the map $\hat{\mathfrak{g}} \to \widetilde{F}_V$ given by $x_i(n) \mapsto (x_i(-1)\,|0\rangle)_{[n]}$ gives an injective homomorphism of Lie algebras. Thus, there is an induced map of enveloping algebras $U_\kappa(\hat{\mathfrak{g}}) \to \widetilde{U}(V_\kappa(\mathfrak{g}))$, which extends by continuity to a map $\psi : \widetilde{U}_\kappa(\hat{\mathfrak{g}}) \to \widetilde{U}(V_\kappa(\mathfrak{g}))$.

From here, it is not difficult to see that these maps are mutually inverse.

First, we claim that ψ is surjective. It suffices to prove that any element of the form $A_{[n]}$, with $A \in V_\kappa(\mathfrak{g})$ and $n \in \mathbb{Z}$ lies in the image. Without loss of generality, we can assume that A has the form

$$A = x_{i_1}(-n_1) \cdots x_{i_k}(-n_k)\,|0\rangle\,.$$

Then, from the relation $Y[C_{-1}D, z] = :Y[C,z]Y[D,z]:$ that we imposed on $\widetilde{U}(V_\kappa(\mathfrak{g}))$, we see that all Fourier coefficients of $Y[A,z]$, in particular, $A_{[n]}$, are infinite sums of products of elements of the form $(x_i(-1)\,|0\rangle)_{[m]}$, which certainly lie in image of ψ. With a little more work (e.g., checking that the "preimage" of this infinite sum is a well-defined element of $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$), it follows that ψ is surjective.

Thus, it remains to prove that $\varphi \circ \psi$ is the identity, and it suffices to check this on a basis of $\hat{\mathfrak{g}}$ in $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$. But then the claim is obvious, since $\varphi((x_i(-1)\,|0\rangle)_{[n]})$ is the nth Fourier coefficient of $X_i(z)$, which is precisely $x_i(-n)$. □

3. The Center of a Vertex Algebra

In the previous section, we showed that the Sugawara–Segal elements are indeed central elements in the completed universal enveloping algebra at the critical level. We are immediately led to ask the question of whether we can obtain any other information about the center of $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$ from the vertex algebra $V_\kappa(\mathfrak{g})$. We first introduce the notion of the center of a vertex algebra. In the case of $V_\kappa(\mathfrak{g})$, these will be the elements whose Fourier coefficients arise from central elements of $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$. Then, we will specialize to the study of the center of $V_\kappa(\mathfrak{g})$ at the critical level. Eventually (not in this talk), we will be able to show that all central elements of $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$ can be obtained from the center of $V_\kappa(\mathfrak{g})$, so we hope to reduce the study of $Z(\widetilde{U}_\kappa(\hat{\mathfrak{g}}))$ to the study of the center of $V_\kappa(\mathfrak{g})$. At the end of this section, we will interpret the center in two ways: first, by embedding it into the ring of $G[[t]]$-invariant endomorphisms on $V_\kappa(\mathfrak{g})$, and then by studying its associated graded structure.

3.1. **Basic Definitions.** The central object of this section is the *center of a vertex algebra*. We want to think of elements in the center as the vectors whose Fourier coefficients produce central elements in $V_\kappa(\mathfrak{g})$. Thus, to motivate our definition of the center, we make the following easy but crucial observation.

Lemma 3.1 (Lemma 3.3.1, [Fre07])**.** *If $A \in V_\kappa(\mathfrak{g})$ satisfies $x_i(m)A = 0$ for all $i = 1, \ldots, \dim \mathfrak{g}$ and $m \geq 0$, then the element $\varphi(A_{[n]}) \in \widetilde{U}_\kappa(\hat{\mathfrak{g}})$ for any $n \in \mathbb{Z}$ is central in $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$.*

Proof. Thanks to Proposition 2.11 and the commutation relations in $\widetilde{F}_V$, we have

$$[x_i(m), \varphi(A_{[n]})] = \sum_{k \geq 0} \binom{m}{k} \varphi((x_i(k)A)_{[m+n-k]})$$

in $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$. Thus, the implication is clear. $\square$

Therefore, we are motivated to give the following definition.

Definition 3.2. The **center of a vertex algebra** V is

$$\mathcal{Z}(V) := \{A \in V \mid B_n A = 0 \quad \forall B \in V, \;\; n \geq 0\}.$$

Equivalently, thanks to associativity,

$$\mathcal{Z}(V) = \{A \in V \mid [Y(A,z), Y(B,z)] = 0 \quad \forall B \in V\}.$$

Observe that $\mathcal{Z}(V)$ is a subspace of V containing the vacuum $|0\rangle$. Moreover, note that $\mathcal{Z}(V)$ is closed under the action of T. Indeed, for any $A \in \mathcal{Z}(V)$, $B \in V$, and $n \geq 0$,

$$B_n(TA) = TB_nA - [T, B_n]A = nB_{n-1}A = 0.$$

Finally, we claim that for any $A, B \in \mathcal{Z}(V)$, the coefficients of $Y(B,z)A$ lie in $\mathcal{Z}(V)$ as well. Indeed, if $A, B \in \mathcal{Z}(V)$ and $C \in V$, we have

$$C_{(n)}B_{(m)}A = B_{(m)}C_{(n)}A + [C_{(n)}, B_{(m)}]A = 0,$$

where we used Equation (1) to deduce $[C_{(n)}, B_{(m)}]A = 0$. These properties suggest that $\mathcal{Z}(V)$ is a "vertex subalgebra," which we shall now define.

Definition 3.3. Let V be a vertex algebra. A subspace $W \subset V$ is a **vertex subalgebra** if

(i) W contains the vacuum $|0\rangle$,
(ii) $TW \subset W$, and
(iii) W is closed under the state-field correspondence, that is, $Y(A,z)B \in W((z))$ for any $A, B \in W$.

Note that the inclusion of a vertex subalgebra into a vertex algebra is a vertex algebra homomorphism. The discussion preceding this definition gives us the following result.

Lemma 3.4 (Lemma 3.3.2, [Fre07])**.** *The center of V is a commutative vertex subalgebra.*

3.2. **Center of the Vacuum Module for Kac–Moody Algebras.** Now, we specialize to the case of the affine Kac–Moody algebra. To begin, we will see that, unlike at the critical level, the center of $V_\kappa(\mathfrak{g})$ does not give rise to any nontrivial central elements in $\widetilde{U}_\kappa(\mathfrak{g})$ away from the critical level. In fact, as we will see in a subsequent talk, the center of $\widetilde{U}_\kappa(\mathfrak{g})$ is trivial away from the critical level, so this result is not surprising.

Proposition 3.5 (Proposition 3.3.3, [Fre07])**.** *The center of $V_\kappa(\mathfrak{g})$ is trivial when $\kappa \neq \kappa_c$.*

Proof. Suppose $A \in V_\kappa(\mathfrak{g})$ is central. We recall the renormalized Sugawara operators which give $V_\kappa(\mathfrak{g})$ the structure of a Vir-module. In particular, $L_0 = \bar{\sigma}_1$ is the grading operator. Yet $\bar{\sigma}_1 A = 0$ since A is central, so it follows that A must be homogeneous of degree zero. But then, A is a scalar multiple of $|0\rangle$. □

Hence, we are more interested in studying $\mathcal{Z}(V_\kappa(\mathfrak{g}))$ at the critical level $\kappa = \kappa_c$. That being said, the results in the remainder of the talk apply for any level κ, though they are far more interesting at the critical level.

Recall (from pg. 4 in Ilya's first set of notes) that, given a commutative associative $\mathbb{C}$-algebra W, we defined a vertex algebra structure on W by setting

$$Y(A,z) = \sum_{n\geq 0} \operatorname{mult}(T^n A)\frac{z^n}{n!}$$

for any $A \in W$. In particular, the endomorphism A_{-1} coincides with the operation of multiplication by A (note the index). Conversely, an easy exercise shows that any commutative vertex algebra V can be given the structure of an associative commutative $\mathbb{C}$-algebra by setting the product AB to $A_{-1}B$ for any $A, B \in V$. In particular, we can give $\mathcal{Z}(V_\kappa(\mathfrak{g}))$ the structure of an associative commutative $\mathbb{C}$-algebra.

Lemma 3.6. *We have an isomorphism of associative algebras $\mathcal{Z}(V_\kappa(\mathfrak{g})) \cong \operatorname{End}_{\widetilde{U}_\kappa(\hat{\mathfrak{g}})} V_\kappa(\mathfrak{g})$.*

Proof. Since $A \in V_\kappa(\mathfrak{g})$ is central if and only if $x_i(n)A = 0$ for all $n \geq 0$, we can identify $\mathcal{Z}(V_\kappa(\mathfrak{g}))$ with the subspace of $\mathfrak{g}[[t]]$-invariant vectors in $V_\kappa(\mathfrak{g})$. Moreover, we have a natural linear isomorphism

$$\operatorname{End}_{\widetilde{U}_\kappa(\hat{\mathfrak{g}})} V_\kappa(\mathfrak{g}) \cong (V_\kappa(\mathfrak{g}))^{\mathfrak{g}[[t]]}$$

given by sending any invariant endomorphism φ to $\varphi(|0\rangle)$. Thus, we have a linear isomorphism $\operatorname{End}_{\widetilde{U}_\kappa(\hat{\mathfrak{g}})} V_\kappa(\mathfrak{g}) \cong \mathcal{Z}(V_\kappa(\mathfrak{g}))$. It remains to show that this isomorphism respects multiplication. Suppose we have invariant endomorphisms φ, ψ. Their composition $\varphi \circ \psi$ is mapped to $\varphi(\psi(|0\rangle))$. Let us write $A := \psi(|0\rangle)$ and $B := \varphi(|0\rangle)$. Then, since $Y(A,z)\,|0\rangle = A + O(z)$, we see that

$$\varphi \circ \psi(|0\rangle) = \varphi(A) = \varphi(A_{-1}\,|0\rangle) = A_{-1}\varphi(|0\rangle) = A_{-1}B,$$

since A_{-1} can be identified with the image of some element in $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$ under the action map. But $A_{-1}B$ is precisely the product of $\varphi(|0\rangle) = A$ and $\varphi(|0\rangle) = B$ in the associative algebra structure on $\mathcal{Z}(V_\kappa(\mathfrak{g}))$. Since $\mathcal{Z}(V_\kappa(\mathfrak{g}))$ is commutative, the result follows. $\square$

From this lemma, we can deduce a result that is not immediately obvious.

Corollary 3.7. *The $\mathbb{C}$-algebra $\operatorname{End}_{\widetilde{U}_\kappa(\hat{\mathfrak{g}})} V_\kappa(\mathfrak{g})$ is commutative. Moreover, away from the critical level, this algebra is one-dimensional.*

Remark 3.8. Since $V_\kappa(\mathfrak{g})$ is a regular module over $U(t^{-1}\mathfrak{g}[t^{-1}])$, we have an injective map $\mathcal{Z}(V_\kappa(\mathfrak{g})) \hookrightarrow U(t^{-1}\mathfrak{g}[t^{-1}])$ which is in fact a map of associative $\mathbb{C}$-algebras.

Remark 3.9. Observe that

$$\widetilde{F}_{\mathcal{Z}(V_\kappa(\mathfrak{g}))} = \operatorname{Span}_{\mathbb{C}} \left\{ \sum_{n\in\mathbb{C}} c_n A_{[n]} \;\middle|\; A \in \mathcal{Z}(V_\kappa(\mathfrak{g})) \right\} \subset \widetilde{F}_{V_\kappa(\mathfrak{g})}$$

forms a central Lie subalgebra of $\widetilde{F}_{V_\kappa(\mathfrak{g})}$. Following the constructions in Section 2.3, we have an inclusion of universal enveloping algebras

$$\widetilde{U}(\mathcal{Z}(V_\kappa(\mathfrak{g}))) \hookrightarrow \widetilde{U}(V_\kappa(\mathfrak{g})).$$

In fact, observe that this inclusion identifies $\widetilde{U}(\mathcal{Z}(V_\kappa(\mathfrak{g})))$ with a subalgebra of the center of $\widetilde{U}(V_\kappa(\mathfrak{g}))$. Thus, through the isomorphism

$$\widetilde{U}(V_\kappa(\mathfrak{g})) \cong \widetilde{U}_\kappa(\hat{\mathfrak{g}})$$

given by Proposition 2.21, we have an injective map

$$\widetilde{U}(\mathcal{Z}(V_\kappa(\mathfrak{g}))) \hookrightarrow Z(\widetilde{U}_\kappa(\hat{\mathfrak{g}})).$$

At the critical level, this map sends the the Sugawara operators, viewed as formal elements $\sigma_{[n+1]} \in \widetilde{F}_{V_\kappa(\mathfrak{g})}$, to the Sugawara elements of $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$. Similar to how we recovered the structure of $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$ purely from the vertex algebra structure on $V_\kappa(\mathfrak{g})$, we can hope to glean information about the center of $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$ purely from the commutative algebra structure on $\mathcal{Z}(V_\kappa(\mathfrak{g}))$.

3.3. **Associated Graded Ring for the Center.** The subspace $\mathcal{Z}(V_\kappa(\mathfrak{g}))$ inherits a filtration from the grading on $V_\kappa(\mathfrak{g})$. We will be interested in studying the associated graded ring $\operatorname{gr} \mathcal{Z}(V_\kappa(\mathfrak{g}))$ with respect to this filtration. In what follows, we write $\mathcal{O} := \mathbb{C}[[t]]$, $\mathfrak{g}_{\mathcal{O}} := \mathfrak{g}[[t]]$, $G(\mathcal{O}) := G[[t]]$, and $\mathfrak{g}^*_{\mathcal{O}} := \mathfrak{g}^*[[t]]$ for readability.

To motivate this study, we recall some facts from the case of a finite-dimensional semisimple Lie algebra $\mathfrak{g}$. In this case, we have an identification

$$\operatorname{gr} Z(U(\mathfrak{g})) \xrightarrow{\sim} \mathbb{C}[\mathfrak{g}^*]^{\mathfrak{g}},$$

given by the natural injective homomorphism $\operatorname{gr}(U(\mathfrak{g})^{\mathfrak{g}}) \hookrightarrow \mathbb{C}[\mathfrak{g}^*]^{\mathfrak{g}}$. This map is an isomorphism due to the $\mathfrak{g}$-module isomorphism $\operatorname{Sym}\mathfrak{g} \to U(\mathfrak{g})$ given by symmetrization:

$$y_1 y_2 \cdots y_m \mapsto \frac{1}{n!} \sum_{\sigma\in S_n} y_{\sigma(1)} y_{\sigma(2)} \cdots y_{\sigma(m)}.$$

We will find an analog to this injective map for $\operatorname{gr} \mathcal{Z}(V_\kappa(\mathfrak{g}))$ in the affine Kac–Moody case, but unlike the finite-dimensional case, this map is not an isomorphism in general.

Proposition 3.10 (Lemma 3.3.1, [Fre07]). *There is an injective graded map of $\mathbb{C}$-algebras*

$$\operatorname{gr} \mathcal{Z}(V_\kappa(\mathfrak{g})) \hookrightarrow \mathbb{C}[\mathfrak{g}_\mathcal{O}^*]^{G(\mathcal{O})},$$

where the action of the loop group $G(\mathcal{O})$ on $\mathbb{C}[\mathfrak{g}_\mathcal{O}^*]$ is induced by its adjoint action on $\mathfrak{g}_\mathcal{O}$.

Proof. Note that the operation of taking the symbol commutes with the action of $\mathfrak{g}_\mathcal{O}$ since the action of $\mathfrak{g}_\mathcal{O}$ cannot increase the degree of any term. Hence, we have an injective $\mathbb{C}$-linear map

$$\operatorname{gr} \mathcal{Z}(V_\kappa(\mathfrak{g})) \hookrightarrow (\operatorname{gr} V_\kappa(\mathfrak{g}))^{\mathfrak{g}_\mathcal{O}} = (\operatorname{gr} V_\kappa(\mathfrak{g}))^{G(\mathcal{O})}.$$

On the other hand, recall the isomorphism

$$V_\kappa(\mathfrak{g}) \cong U(t^{-1}\mathfrak{g}[t^{-1}]).$$

By the Poincaré–Birkhoff–Witt theorem, there is an isomorphism of $\mathbb{C}$-algebras

$$\operatorname{gr} U(t^{-1}\mathfrak{g}[t^{-1}]) \cong \operatorname{Sym}(t^{-1}\mathfrak{g}[t^{-1}]).$$

This isomorphism also intertwines the action of $G(\mathcal{O})$, so altogether, we are left with an injective map

$$\operatorname{gr} \mathcal{Z}(V_\kappa(\mathfrak{g})) \hookrightarrow \operatorname{Sym}(t^{-1}\mathfrak{g}[t^{-1}])^{G(\mathcal{O})} \subset \operatorname{Sym}(t^{-1}\mathfrak{g}[t^{-1}]).$$

The fact that this map is a $\mathbb{C}$-algebra homomorphism follows since the injective map

$$\mathcal{Z}(V_\kappa(\mathfrak{g})) \hookrightarrow U(t^{-1}\mathfrak{g}[t^{-1}])$$

from Remark 3.8 is a $\mathbb{C}$-algebra homomorphism.

Finally, recall that the symmetric algebra $\operatorname{Sym}(t^{-1}\mathfrak{g}[t^{-1}]) = \operatorname{Sym}((\mathfrak{g}((t))/\mathfrak{g}_\mathcal{O})^*)$ is the space of polynomial functions on the topological dual space $(\mathfrak{g}((t))/\mathfrak{g}_\mathcal{O})^*$. This dual space can be identified with the space of one-forms $\mathfrak{g}_\mathcal{O}^*\, dt$, where an element $\phi(t)\, dt \in \mathfrak{g}_\mathcal{O}^*$ acts on $A(t) \in \mathfrak{g}((t))$ by the pairing

$$\langle \phi(t)\, dt, A(t)\rangle = \operatorname{Res}_{t=0}\langle \phi(t), A(t)\rangle\, dt.$$

The adjoint action of $G(\mathcal{O})$ on $\mathfrak{g}_\mathcal{O}$ induces an action on $\mathbb{C}[\mathfrak{g}_\mathcal{O}^*]$, which coincides with the induced action of $G(\mathcal{O})$ on $\operatorname{Sym}(t^{-1}\mathfrak{g}[t^{-1}])$. Thus, we have further identifications

$$\operatorname{Sym}(t^{-1}\mathfrak{g}[t^{-1}])^{G(\mathcal{O})} = \mathbb{C}[(\mathfrak{g}((t))/\mathfrak{g}_\mathcal{O})^*]^{G(\mathcal{O})} = \mathbb{C}[\mathfrak{g}_\mathcal{O}^*]^{G(\mathcal{O})},$$

proving the proposition. □

3.4. **Invariant Functions.** Our goal now is to bound the size of $\mathcal{Z}(V_\kappa(\mathfrak{g}))$ at the critical level by studying the algebra of $\mathfrak{g}_\mathcal{O}$-invariant functions in $\mathbb{C}[\mathfrak{g}_\mathcal{O}^*]$.

The finite-dimensional analog to $\mathbb{C}[\mathfrak{g}_\mathcal{O}^*]^{G(\mathcal{O})}$ is the algebra $\mathbb{C}[\mathfrak{g}^*]^\mathfrak{g} = \mathbb{C}[\mathfrak{g}^*]^G$ from before. This latter algebra is a finitely generated polynomial algebra. Let us write

$$\mathbb{C}[\mathfrak{g}^*]^G = \mathbb{C}[f_1, \ldots, f_r],$$

where each r is the rank of $\mathfrak{g}$ and each f_i is a homogeneous generator of degree d_i, where the collection $\{d_i\}$ forms the set of degrees of $\mathfrak{g}$. For example, when $\mathfrak{g} = \mathfrak{sl}_n$, each f_i corresponds to the map $[x \mapsto \operatorname{tr}(x^i)]$.

We will produce an analogous result for $\mathbb{C}[\mathfrak{g}^*_{\mathcal{O}}]^{G(\mathcal{O})}$ in the affine Kac–Moody case. In particular, we will use the polynomials f_i to produce elements in $\mathbb{C}[\mathfrak{g}^*_{\mathcal{O}}]^{G(\mathcal{O})}$. This construction will come from the following general observation.

For any associative $\mathbb{C}$-algebra R, consider the group $G(R)$ formed by the R-points of G. Let $\mathfrak{g}_R := \mathfrak{g} \otimes_{\mathbb{C}} R$ and $\mathfrak{g}^*_R := \mathfrak{g}^* \otimes_{\mathbb{C}} R$. Then, each G-invariant polynomial $f \in \mathbb{C}[\mathfrak{g}^*]^G$ defines a manifestly $G(R)$-invariant map $f_R : \mathfrak{g}^*_R \to R$.

We consider the particular case of $R = \mathcal{O}$. Any polynomial map $f \in \mathbb{C}[\mathfrak{g}^*]^G$ defines a $G(\mathcal{O})$-invariant map $f_{\mathcal{O}} : \mathfrak{g}^*_{\mathcal{O}} \to \mathcal{O}$. For each $n \geq 0$, we obtain a $G(\mathcal{O})$-invariant polynomial map $f_n : \mathfrak{g}^*_{\mathcal{O}} \to \mathbb{C}$ by pairing with the one-form $t^{-n-1}\,dt \in \mathbb{C}[t^{-1}]\,dt$ and taking the residue at $t = 0$:

$$f_n(x(t)) := \mathrm{Res}_{t=0}\, f_{\mathcal{O}}(x(t))t^{-n-1}\,dt \in \mathbb{C}[\mathfrak{g}^*_{\mathcal{O}}]^{G(\mathcal{O})}.$$

In other words, the polynomials $f_n(x(t))$ are the coefficients in the power series

$$f_{\mathcal{O}}(x(t)) = \sum_{n\geq 0} f_n(x(t))t^n \in \mathcal{O}[\mathfrak{g}^*_{\mathcal{O}}].$$

Recall the generators $f_1, \ldots, f_r$ for $\mathbb{C}[\mathfrak{g}^*]^G$. Applying the construction above to these polynomials yields $G(\mathcal{O})$-invariant polynomials $f_{i,n}(x(t)) \in \mathbb{C}[\mathfrak{g}^*_{\mathcal{O}}]^{G(\mathcal{O})}$ for $i = 1, \ldots, r$ and $n \geq 0$. In analogy with the finite-dimensional case, we have the following result.

Proposition 3.11. *The polynomials $f_{i,n}(x(t))$ generate $\mathbb{C}[\mathfrak{g}^*_{\mathcal{O}}]^{G(\mathcal{O})}$.*

Proof. Postponed to the subsequent lectures. $\square$

We conclude with an explicit calculation.

Example 3.12. Consider the case $G = \mathrm{SL}_n$ and $\mathfrak{g} = \mathfrak{sl}_n$. Recall that the algebra of invariant functions $\mathbb{C}[\mathfrak{g}^*]^G$ is the polynomial algebra generated by $f_i := [x \mapsto \mathrm{tr}(x^i)]$ for $i = 1, \ldots, n-1$. For each $i = 1, \ldots, n-1$, the map $(f_i)_{\mathcal{O}}$ defined above is given by the map $[x(t) \mapsto \mathrm{tr}(x(t)^i)]$. Observe that this map is clearly $G(\mathcal{O})$-invariant: given $g(t) \in G(\mathcal{O})$, we have

$$[g(t) \cdot (f_i)_{\mathcal{O}}](x(t)) = \mathrm{tr}(g(t)x(t)^i g(t)^{-1}) = \mathrm{tr}(x(t)^i).$$

Example 3.13. Let us specialize further to the case $G = \mathrm{SL}_2$ and $\mathfrak{g} = \mathfrak{sl}_n$. We will explicitly calculate the generators $f_{i,n}$ in this case. Fix the standard basis $\{e, f, h\}$ for $\mathfrak{g}$. In this case, $\mathbb{C}[\mathfrak{g}^*]^{\mathrm{SL}_2}$ is the polynomial algebra $\mathbb{C}[\Omega]$ generated by the quadratic Casimir

$$\Omega(e, f, h) = \frac{1}{2}(ef + fe + h^2).$$

Write

$$E(t) = \sum_{n<0} e(n)t^{-n-1}, \quad F(t) = \sum_{n<0} f(n)t^{-n-1}, \quad H(t) = \sum_{n<0} h(n)t^{-n-1}.$$

Following the construction above, we have the $G(\mathcal{O})$-invariant map

$$\begin{aligned}\Omega_{\mathcal{O}}(E(t), F(t), H(t)) &= \frac{1}{2}(E(t)F(t) + F(t)E(t) + H(t)^2) \\ &= \frac{1}{2}\sum_{n<0}\left(\sum_{\substack{m,\ell<0\\ m+\ell=n}} e(m)f(\ell) + f(\ell)e(m) + h(\ell)h(m)\right) t^{-n-2}.\end{aligned}$$

The coefficients of $\Omega_{\mathcal{O}}$ are

$$\Omega_n := \frac{1}{2}\sum_{\substack{m,\ell<0\\ m+\ell=n}} e(m)f(\ell) + f(\ell)e(m) + h(\ell)h(m) \in \mathbb{C}[\mathfrak{g}^*_{\mathcal{O}}]^{G(\mathcal{O})}.$$

This expression is the symbol of $S_n\,|0\rangle$, where S_n is the nth Sugawara operator.

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