HECKE ALGEBRAS AND KAZHDAN-LUSZTIG BASIS

1. K_0 AND CHARACTERS

As before, $\mathfrak{g}$ is a semisimple Lie algebra and $\mathcal{O}_{W\cdot 0}$ is the principal block of category $\mathcal{O}$. For $M \in \mathcal{O}$, we can consider its *character*: $\operatorname{ch} M = \sum_{\chi\in\Lambda} \dim M_\chi e^\chi$.

Example 1.1. *Since* $\Delta(\lambda) \cong U(\mathfrak{n}^-)\otimes\mathbb{C}_\lambda$ *as a* $U(\mathfrak{b}_-)$*-module, we have* $\operatorname{ch}\Delta(\lambda) = e^\lambda \prod_{\alpha\in\Delta_+}(1-e^{-\alpha})^{-1}$.

Exercise 1.2. *More generally, for a parabolic Verma module, we have*

$$\operatorname{ch}\Delta_J(\lambda) = \frac{\sum_{w\in W_J}(-1)^{\ell(w)}e^{w(\lambda+\rho)}}{\sum_{w\in W_J}(-1)^{\ell(w)}e^{w\rho}} \prod_{\alpha\in\Delta_+\setminus\Delta_J}(1-e^{-\alpha})^{-1}.$$

The question is to compute the characters of more interesting modules: $L(\lambda), P(\lambda), T(\lambda)$. The translation functors reduce these questions to the principal block $\mathcal{O}_{W\cdot 0}$. This is easy for $L(\lambda)$'s, the case of $P(\lambda)$'s can be deduced from there using the BGG reciprocity. And, in fact, the case of $T(\lambda)$'s can also be done (using Ringel duality).

Consider the Grothendieck group $K_0(\mathcal{O}_{W\cdot 0})$. It is a free $\mathbb{Z}$-module with basis formed by $[L(w\cdot 0)]$.

Exercise 1.3. *Prove that each of the following collections form a basis in* $K_0(\mathcal{O}_{W\cdot 0})$*:*

(1) $[\Delta(\lambda)]$ *for* $\lambda\in W\cdot 0$,
(2) $[P(\lambda)]$ *for* $\lambda\in W\cdot 0$,
(3) $[T(\lambda)]$ *for* $\lambda\in W\cdot 0$.

Hint: use triangularity.

Our initial basis will be that of Vermas, because their characters are easy to compute. And since the characters are additive on K_0, to compute the character of $M\in\mathcal{O}_{W\cdot 0}$ it is enough to express $[M]$ as a linear combination of the classes $[\Delta(\lambda)]$.

Note that we can identify $K_0(\mathcal{O}_{W\cdot 0})$ with $\mathbb{Z}W$. In fact, there are two similar – but different – ways to do that: via $w\mapsto[\Delta(w\cdot 0)]$ (convenient for working with $[P(\lambda)]$ because $P(0)=\Delta(0)$) or via $w\mapsto[\Delta(w\cdot(-2\rho))]$ (good for $[L(\lambda)]$'s and $[T(\lambda)]$'s because $L(-2\rho)=\Delta(-2\rho)=T(-2\rho)$). We note that under any of these identifications, Θ_i acts on K_0 by $w\mapsto w(1+s_i)$.

However, it turns out that the interesting bases cannot be described entirely on the level of $\mathbb{Z}W$. But they can be described via the *Hecke algebra*, a deformation of $\mathbb{Z}W$ and have to do with the so called *Kazhdan-Lusztig* basis, which is the main object for this lecture.

2. Kazhdan-Lusztig theory

2.1. **Hecke algebras of Coxeter groups.** Let W be a Coxeter group with set S of simple reflections (e.g., a Weyl group of a semisimple Lie algebra). On W, we have the Bruhat order. Also, we can consider the length function $\ell : W \to \mathbb{Z}_{\geqslant 0}$. Recall that for $s \in S, w \in W$, we get $\ell(sw) = \ell(w) - 1$ if w has a reduced expression that starts with s and $\ell(sw) = \ell(w) + 1$ else.

Set $\mathcal{L} := \mathbb{Z}[v, v^{-1}]$. Our goal is to define a deformation of the group ring $\mathbb{Z}W$ over $\mathcal{L}$.

Definition 2.1. *The Hecke algebra $\mathcal{H} = \mathcal{H}(W, S) = \bigoplus_{x\in W} \mathcal{L}H_x$ is the free $\mathcal{L}$-module with basis $H_w, w \in W$, and an associative product that is uniquely determined by*

$$H_xH_y = H_{xy} \text{ if } \ell(xy) = \ell(x) + \ell(y),$$
$$H_xH_s = H_{xs} + (v^{-1} - v)H_x \text{ if } \ell(xs) = \ell(x) - 1.$$

The basis H_x is usually called the *standard basis.*

It is a nontrivial fact that such a product exists. Note that $\mathcal{H}/(v - 1) = \mathbb{Z}W$ so we indeed get a deformation.

Exercise 2.2. *Prove the following facts about $\mathcal{H}$:*

1. *H_1 is a unit.*
2. *The elements $H_s, s \in S$, are generators.*
3. *The relations for these generators are as follows: $(H_s - v^{-1})(H_s + v) = 0$ for all $s \in S$ and $H_sH_tH_s \ldots = H_tH_sH_t \ldots$ for all $s \neq t \in S$, where in both sides we have m_{st} factors, m_{st} is the order of st in W.*
4. *$H_sH_x = H_{sx} + (v^{-1} - v)H_x$ if $\ell(sx) = \ell(x) - 1$.*
5. *The assignments $H_x \mapsto v^{-\ell(w)}, H_x \mapsto (-v)^{\ell(w)}$ define $\mathcal{L}$-linear representations of $\mathcal{H}$ in $\mathcal{L}$, called the trivial and sign representations, denote them by triv_v and sgn_v.*

2.2. **Bar involution and Kazhdan-Lusztig basis.**

Lemma 2.3 (Bar involution)**.** *There exists a unique ring involution $a \mapsto \bar{a} \colon \mathcal{H} \to \mathcal{H}$, $H \mapsto \overline{H}$ such that $\overline{v} = v^{-1}$ and $\overline{H_x} = (H_{x^{-1}})^{-1}$.*

Proof. Exercise. □

Definition 2.4. *We call $a \in \mathcal{H}$ self-dual if $\overline{a} = a$.*

Theorem 2.5 (Kazdan-Lusztig basis)**.** *For all $x \in W$ there exists a unique self-dual element $\underline{H}_x \in \mathcal{H}$ such that $\underline{H}_x \in H_x + \sum_y v\mathbb{Z}[v]H_y$. Moreover, we have $\underline{H}_x \in H_x + \sum_{y \prec x} v\mathbb{Z}[v]H_y$.*

Proof. We prove the existence and the uniqueness is an exercise.

Note that $H_1 = 1$. Set $C_s = H_s + v$. We see that $\overline{C}_s = H_s^{-1} + v^{-1} = H_s + v = C_s$. So $\underline{H}_s = C_s$. We have $H_x C_s = H_{sx} + v^{\ell(xs)-\ell(x)} H_x$ (note that $\ell(xs) - \ell(x) \in \{\pm 1\}$).

We prove the existence of $\underline{H}_x$ by induction on $\ell(x)$, the case of $\ell(x) = 1$ is already done. Take $s \in S$ such that $\ell(xs) = \ell(x) - 1$. We have already constructed $\underline{H}_{xs}$. Note that $\underline{H}_{xs} \in H_{xs} + \sum_{y \prec sx} v\mathbb{Z}[v] H_y$ by our choice of s so $\underline{H}_{xs} C_s \in H_s + \sum_{y \prec x} \mathbb{Z}[v] H_y$. So we can write $\underline{H}_{xs} C_s = H_x + \sum_{y \prec x} h_y H_y$ for some $h_y \in \mathbb{Z}[v]$. The element $\underline{H}_{xs} C_s$ is self-dual as a product of self-dual elements but we still can have $h_y(0) \neq 0$, so we cannot take $\underline{H}_{xs} C_s$ for $\underline{H}_x$. Instead, we set $\underline{H}_x := \underline{H}_{xs} C_s - \sum_y h_y(0) \underline{H}_y$. □

Example 2.6. *Consider the case $W = S_3 = \langle s_1, s_2 \rangle$, where $s_1 = (12)$, $s_2 = (23)$. We have $\underline{H}_1 = 1$, $\underline{H}_{s_1} = C_{s_1}$, $\underline{H}_{s_2} = C_{s_2}$. We see that*

$$\underline{H}_{s_1 s_2} = C_{s_1} C_{s_2} = T_{s_1 s_2} + v(T_{s_1} + T_{s_2}) + v^2 T_{s_1 s_2}.$$

and, similarly, $\underline{H}_{s_2 s_1} = C_{s_2} C_{s_1}$. It remains to compute $\underline{H}_{s_1 s_2 s_1}$. We have $C_{s_1} C_{s_2} C_{s_1} = H_{s_1 s_2 s_1} + v H_{s_1 s_2} + v H_{s_2 s_1} + v^2 H_{s_1} + v^2 H_{s_2} + H_{s_1} + v^3 + v$. We should now substract C_s and get

$$\underline{H}_{s_1 s_2 s_1} = H_{s_1 s_2 s_1} + v(H_{s_1 s_2} + H_{s_2 s_1}) + v^2(H_{s_1} + H_{s_2}) + v^3.$$

Let us list some properties of the elements $\underline{H}_x$.

Example 2.7. *Let W be finite so that it makes sense to speak about the longest element w_0. We have $\underline{H}_{w_0} = \sum_{u \in W} v^{\ell(w_0) - \ell(u)} H_u =: R$.*

Proof. Note that $RC_s = (v + v^{-1})R$, hence, $RH_s = v^{-1}R$ and $R\mathcal{H} \simeq \mathrm{triv}_v$. We also have $\overline{R}C_s = (v + v^{-1})\overline{R}$. It easily follows that $\overline{R} \in \mathcal{L}R$. Note now that $R \in \underline{H}_{w_0} + \sum_{y \prec w_0} \mathcal{L} \underline{H}_y$ so $R = \overline{R}$. □

Exercise 2.8. *Let $J \subset I$. Then $\mathcal{H}(W_J) \hookrightarrow \mathcal{H}(W)$ via $H_x \mapsto H_x$ for $x \in W_J$. Show that this embedding sends $\underline{H}_x$ to $\underline{H}_x$.*

Definition 2.9. *For $x, y \in W$ we define the* Kazhdan-Lusztig polynomial $h_{y,x} \in \mathbb{Z}[v]$ *by $\underline{H}_x = \sum_y h_{y,x}(v) H_y$.*

Exercise 2.10. *For any $x, y \in W$, we have $h_{y,x} = h_{y^{-1}, x^{-1}}$.*

Proof. Use the anti-authomorphism $v \mapsto v, H_x \mapsto H_{x^{-1}}$. □

Here is another version of the Kazhdan-Lusztig basis.

Theorem 2.11 (c.f. Theorem 2.5)**.** *For all $x \in W$ there exists a unique self-dual $\underline{\tilde{H}}_x \in \mathcal{H}$ such that $\underline{\tilde{H}}_x = H_x + \sum_y v^{-1}\mathbb{Z}[v^{-1}] H_y$. We have $\underline{\tilde{H}}_x = H_x + \sum_y h_{y,x}(-v^{-1}) H_y$.*

This is because of the ring involution of $\mathcal{H}$ that fixes all H_x and sends v to $-v^{-1}$.

2.3. **Application: multiplicities in category $\mathcal{O}$.** Let us return to the setting of the first section of this lecture.

The following statement is usually called a *Kazhdan-Lusztig* (type) theorem. The part about simples (which is equivalent to the part about projectives thanks to the BGG reciprocity and some combinatorics) was originally conjectured by Kazhdan-Lusztig and then proved by Beilinson-Bernstein and Brylinski-Kashiwara.

Theorem 2.12. *We have the following equalities in $K_0(\mathcal{O}_{W\cdot 0})$:*

(1) $[P(x\cdot 0)] = \sum_{y\in W} h_{y,x}(1)[\Delta(y\cdot 0)]$, *equivalently, if we send $\Delta(x\cdot 0)$ to $H_x|_{v=1}$, then $[P(x\cdot 0)]$ becomes $\underline{H}_x|_{v=1}$.*

(2) $[L(x\cdot(-2\rho))] = \sum_{y\in W} h_{y,x}(-1)[\Delta(y\cdot(-2\rho))]$,

(3) $[T(x\cdot(-2\rho))] = \sum_{y\in W} h_{y,x}(1)[\Delta(y\cdot(-2\rho))]$.

3. Variations: spherical and anti-spherical modules

It turns out that the Kazhdan-Lusztig basis in $\mathcal{H}$ gives rise to similar bases in certain modules.

We fix a subset $S_J \subset S$ and the corresponding Coxeter group $W_J \subset W$ and denote by $W^J \subset W$ the set of all $w \in W$ such that w has minimal length in $W_J w$. So we have a bijection $W_J \times W^J \xrightarrow{\sim} W$, $(x, y) \mapsto xy$. Set $\mathcal{H}_J := \mathcal{H}(W_J)$ and consider the induced right modules

$$\mathcal{M}(=\mathcal{M}^J) := \mathrm{triv}_J \otimes_{\mathcal{H}_J} \mathcal{H},\ \mathcal{N}(=\mathcal{N}^J) = \mathrm{sgn}_J \otimes_{\mathcal{H}_J} \mathcal{H}.$$

These are the *spherical* and *anti-spherical* modules, respectively. We have the *standard bases* $M_x = 1\otimes H_x \in \mathcal{M}$ and $N_x = 1\otimes H_x \in \mathcal{N}$, where $x \in W^J$. The bar involution on $\mathcal{H}$ induces compatible involutions on $\mathcal{M}$ and $\mathcal{N}$ by $\overline{1\otimes a} = 1\otimes \bar{a}$, $a \in \mathcal{H}$.

Exercise 3.1. *These are well-defined.*

Theorem 3.2. *For all $x \in W^J$, there exists a unique self-dual $\underline{M}_x \in \mathcal{M}$ such that $\underline{M}_x \in M_x + \sum_y v\mathbb{Z}[v]M_y$. The same for $\mathcal{N}$.*

Exercise 3.3. *Check that the proof of Theorem 2.5 carries over to this case.*

Definition 3.4. *For $x, y \in W^J$ we define $m_{y,x} \in \mathbb{Z}[v]$ by $\underline{M}_x = \sum_y m_{y,x}M_y$. Define $n_{y,x}$ similarly. Then $m_{y,x}, n_{y,x}$ are called parabolic Kazhdan-Lusztig polynomials.*

Let us now describe the relation between the parabolic and the ordinary Kazhdan-Lusztig polynomials.

Proposition 3.5. *Suppose W_J is finite. Then*

(1) *If $w_{0,J} \in W_J$ denotes the longest element then we have $m_{y,x} = h_{w_{0,J}y, w_{0,J}x}$.*

(2) $n_{y,x} = \sum_{z\in W_J} (-v)^{l(z)} h_{zy,x}$.

Proof. We have the embedding $\iota\colon \mathcal{M} \hookrightarrow \mathcal{H}$ of right $\mathcal{H}$-modules via $1 \otimes a \mapsto \underline{H}_{w_{0,J}} a = (\sum_{w \in W_J} v^{\ell(w_{0,J})-\ell(w)} H_w) a$, denote it by ι. So $\iota(\underline{M}_x)$ is self dual and it lies in $H_{w_{0,J}x} + v\,\mathrm{Span}_{\mathbb{Z}[v]}(H_y, y \in W)$. By the uniqueness part of Theorem 2.5, $\iota(\underline{M}_x) = \underline{H}_{w_f x}$. This easily implies (1).

The proof of (2) is similar and is based on the canonical surjection $\sigma\colon \mathcal{H} \twoheadrightarrow \mathcal{N}, H \mapsto 1 \otimes H$ that can be shown to map $\underline{H}_x$ to $\underline{N}_x$ for $x \in W^J$. $\square$

We finish with the following theorem (c.f. Theorem 2.11).

Theorem 3.6. *For all $x \in W^J$ there exists a unique self-dual $\underline{\tilde{N}}_x \in \mathcal{N}$ such that $\underline{\tilde{N}}_x \in N_x + \sum_{y \prec x} v^{-1}\mathbb{Z}[v^{-1}] N_y$. The similar claim holds for $\mathcal{M}$.*

3.1. **Representation theoretic relevance.** We return to the representation theoretic setting, in particular, our W is the Weyl group of $\mathfrak{g}$. Consider the corresponding parabolic category $\mathcal{O}_{W \cdot 0, J}$.

Let us consider its Grothendieck group of the parabolic category. Note that $x \cdot 0 \in \Lambda_J^+ \Leftrightarrow x \in W^J$ and $y \cdot (-2\rho) \in \Lambda_J^+ \Leftrightarrow y \in w_{0,J} W^J$.

Exercise 3.7. *In $K_0(\mathcal{O}_{W\cdot 0})$, we have $[\Delta_J(w \cdot 0)] = \sum_{w \in W_J} (-1)^{\ell(w)} [\Delta(yw \cdot 0)]$ for $w \in W^J$.*

So we can identify $K_0(\mathcal{O}_{W\cdot 0, J})$ with both $\mathcal{M}|_{v=-1}$ and $\mathcal{N}|_{v=1}$ so that $[\Delta_J(x \cdot 0)]$ gets identified with $M_x|_{v=-1}$ in the first case and $N_x|_{v=1}$ in the second case. This is the identification of right W-modules, where the action of s_i on the K_0 is via $[\Theta_i] + 1$.

Theorem 3.8 (Parabolic Kazhdan-Lusztig theorem)**.** *The following claims are true:*

(1) $[P_J(x \cdot 0)] = \underline{N}_x|_{v=1}$ *if we identify* $[\Delta_J(x \cdot 0)] = N_x|_{v=1}$.
(2) $[L(w_{0,J}x \cdot (-2\rho))] = \underline{M}_x|_{v=-1}$ *if we identify* $[\Delta_J(w_{0,J}x \cdot (-2\rho))] = M_x|_{v=-1}$.
(3) $[T_J(w_{0,J}x \cdot (-2\rho))] = \underline{N}_x|_{v=1}$ *if we identify* $[\Delta_J(w_{0,J}x \cdot (-2\rho))] = N_x|_{v=1}$..