Vertex algebras

02/08/24

0. Motivation:

Take $k \in \mathbb{C}$. $\quad 0 \to \mathbb{C}\mathbb{1} \to \hat{\mathfrak{g}}_k \to \mathfrak{g}((t)) \to 0$.

($\mathbb{1}$ is central)

$$[xf(t), yg(t)] = [x,y]f(t)g(t) - (k(x,y)\,\mathrm{Res}\, f dg)\mathbb{1}$$

invariant form

let $U_k(\hat{\mathfrak{g}}) = U(\hat{\mathfrak{g}}_k) / (\mathbb{1}-1)$

We call a $U_k(\hat{\mathfrak{g}})$-module M <u>smooth</u> if $\forall\, v \in M$ $\quad (\mathfrak{g} \otimes t^N \mathfrak{g}[[t]]) \cdot v = 0$ for $N \gg 0$.

All modules from category $\mathcal{O}$ are smooth, and we will be interested in smooth modules only. So, it's natural to consider

$$\tilde{U}_k(\hat{\mathfrak{g}}) = \varprojlim U_k(\hat{\mathfrak{g}}) / I_N,$$

where $I_N = U_k(\hat{\mathfrak{g}}) \cdot (\mathfrak{g} \otimes t^N \mathbb{C}[[t]])$

(I_N is a left ideal, but the limit is an algebra!)

<u>Question</u>: Describe the center $Z(\tilde{U}_k(\hat{\mathfrak{g}}))$.

<u>„Answer"</u>:

Trivial for all $k \neq k_c \in \mathbb{C}$ (for $\mathfrak{g} = \mathfrak{sl}_n$, and $(x,y) = \mathrm{tr}(xy)$, $k_c = -n$)

Large (and interesting) for $k = k_c$

It turns out, it is easier to find the center not of the associative algebra $\tilde{\mathcal{U}}_{k_c}(\hat{\mathfrak{g}})$, but of the related vertex algebra $V_{k_c}(\hat{\mathfrak{g}})$ (and $Z(\tilde{\mathcal{U}}_{k_c}(\hat{\mathfrak{g}}))$ can be recovered from $Z(V_{k_c}(\hat{\mathfrak{g}}))$).

---

1. Basics of V.A.'s:

Consider $\mathbb{C}[[z^{\pm 1}, w^{\pm 1}]] = \left\{ \sum_{i,j \in \mathbb{Z}} \alpha_{ij} z^i w^j \right\}$.

Define $\delta(z-w) \in \mathbb{C}[[z^{\pm 1}, w^{\pm 1}]]$,

$$\delta(z-w) := \sum_{n \in \mathbb{Z}} z^n w^{-n-1}$$

(note that elements of $\mathbb{C}[[z^{\pm 1}, w^{\pm 1}]]$ can not be multiplied in general, but they can be multiplied by elements of $\mathbb{C}[z^{\pm 1}, w^{\pm 1}]$.

Exercise:

(1) $A(z)\,\delta(z-w) = A(w)\,\delta(z-w) \quad \forall A \in \mathbb{C}[[z^{\pm 1}]]$

(2) $(z-w)\,\delta(z-w) = 0$

(3) $(z-w)^{n+1}\,\partial_w^n\,\delta(z-w) \quad \left(\partial_w = \frac{\partial}{\partial w}\right).$

Let V - vector space.

Def. A <u>field</u> is $A(z) = \sum_{n\in\mathbb{Z}} A_n z^{-n-1} \in (\mathrm{End}\, V)[[z^{\pm 1}]]$, s.t. $\forall\, v \in V$ $A_n v = 0$ $\forall\, n \gg 0 \iff$ $A(z)v \in V((z))$ $\forall\, v \in V$.

Def <u>Vertex algebra</u> is $(V, |0\rangle, T, Y(\cdot, z))$:

(1) A vector space V. (space of states)

(2) $|0\rangle \in V$ (vacuum vector)

(3) $T: V \to V$ (translation operator)

(4) $Y(\cdot, z): V \to \mathrm{End}[[z^{\pm 1}]]$, (state-field correspondence) s.t.

$\forall\, A \in V$ $\quad Y(A, z) = \sum_{n\in\mathbb{Z}} A_{(n)} z^{-n-1}$ is a field

with following axioms:

(i) $Y(|0\rangle, z) = \mathrm{id}_V$

(ii) $Y(A, z)|0\rangle = A + z(\ldots) \in V[[z]]$.

(iii) $[T, Y(A, z)] = \partial_z Y(A, z)$

(iv) $T|0\rangle = 0$.

(v) (locality) $\forall\, A, B \in V$ $\exists\, N \in \mathbb{Z}_{\geq 0}$ s.t.

$$(z-w)^N [Y(A,z), Y(B,w)] = 0$$

It follows from (ii), (iii) that $T(A) = A_{(-2)}|0\rangle$.

Def. We say V is graded if $V = \bigoplus_{n\in\mathbb{Z}} V_n$, $|0\rangle \in V_0$, $\deg T = 1$, and $\forall A \in V_m$,

$$\deg A_{(n)} = -n+m+1.$$

Example (commutative V.A.)

Let V - commutative associative unital algebra with a derivation T.

Define:

$$Y(A,z) := \sum_{n\geq 0} (T^n(A)) \frac{z^n}{n!} = \exp(Tz)\cdot A$$

It is clear, all axioms (i)-(V) are satisfied, and, moreover, in (V) we can take $N=0$:

$$(*) \quad [Y(A,z), Y(B,w)] = 0.$$

Def. A V.A. with property (*) is called commutative.

Lemma. V is commutative $\iff$ $Y(A,z) \in (\mathrm{End}\, V)[[z]]$ $\forall A \in V$.

$\square$ $\Rightarrow$: we have $\forall A, B$:

$$Y(A,z)Y(B,w)|0\rangle = Y(B,w)Y(A,z)|0\rangle.$$

Take coefficients of w^0. Using (iii), get:

$Y(A,z)B \in V[[z]]$. $\forall A, B$, as claimed.

$\Leftarrow$: $\forall A, B$, $\exists N$:

$$(z-w)^N [Y(A,z), Y(B,w)] = 0 \in (\mathrm{End}\, V)[[z,w]].$$

But $(z-w)$ is not a zero divisor in this ring! So may take $N=0$ $\square$.

One can further see that any comm. V.A. arises from an algebra in this way, and we have an equivalence of categories:

Commutative V.A. $\iff$ comm. assoc. unital algebras with derivation

② V.A., associated with $\hat{\mathfrak{g}}$.

Let $\mathfrak{g}$ be a simple Lie algebra, let $\{J^a\}$, $a=1,\dots,\dim\mathfrak{g}$ be its basis.

Denote $J^a_n = J^a t^n \in \hat{\mathfrak{g}}$, $a=1,\dots,\dim\mathfrak{g}$.

Then J^a_n, $\mathbb{1}$ $(n\in\mathbb{Z})$ is a (topological) basis of $\hat{\mathfrak{g}}$. It's natural to define

$$J^a(z) = \sum_{n\in\mathbb{Z}} J^a_n z^{-n-1} \in \tilde{\mathcal{U}}_k(\hat{\mathfrak{g}})[[z^{\pm 1}]].$$

We want to define a V.A. $V_k(\mathfrak{g})$, and $J^a(z)$ should be vertex operators in it.

So, there is $|0\rangle \in V_k(\mathfrak{g})$, and $J^a(z)|0\rangle \in V_k(\mathfrak{g})[[z]]$, that is $J^a_n|0\rangle = 0$ for $n\geq 0$.

This leads to consideration of:

$\mathfrak{g}[[t]] \dot{\oplus} \mathbb{C}\mathbb{1} \subset \hat{\mathfrak{g}}$ - a (parabolic) subalgebra.

Let $\mathbb{C}_k$ be its 1-dimensional rep'n: $\mathfrak{g}[[t]]$ acts trivially, $\mathbb{1}$ acts by 1.

As a vector space:

$$V_k(\mathfrak{g}) = \mathcal{U}(\hat{\mathfrak{g}}) \underset{\mathcal{U}(\mathfrak{g}[[t]]\oplus\mathbb{C}\mathbb{1})}{\otimes} \mathbb{C}_k$$

called the <u>vacuum module</u>. Denote by $|0\rangle$ its highest vector.

By PBW-theorem, $V_k(\mathfrak{g})$ has a basis:

$$J^{a_1}_{n_1} \cdots J^{a_m}_{n_m} |0\rangle,$$

$n_1 \le \dots \le n_m < 0$, and if $n_i = n_{i+1}$, then $a_i \le a_{i+1}$, and $\mathrm{gr}_{PBW}(V_k(\mathfrak{g})) = S^{\bullet}(t^{-1}\mathfrak{g}[t^{-1}])$.

Define the grading by $\deg J^a_n = -n$, $\deg|0\rangle = 0$.

Define $T = -\partial_t : V_k(\mathfrak{g}) \to V_k(\mathfrak{g})$, that is:

$$[T, J^a_n] = -n J^a_{n-1},$$

$$T|0\rangle = 0.$$

We are left to define $Y(\cdot, z)$.

Idea: we want $\mathrm{gr}\, V_k(\mathfrak{g})$ with the structure of commutative V.A. to be the degeneration of $V_k(\mathfrak{g})$. So, in $\mathrm{gr}\, V_k(\mathfrak{g})$:

$$Y(\overline{J}^a_{-1}, z) = \sum_{n\ge 0} (T^n(\overline{J}^a_{-1})) \frac{z^n}{n!} = \sum_{n \ge 0} \overline{J}^a_{-n-1} z^n.$$

This leads us to define:

$$Y(J^a_{-1}|0\rangle, z) = \sum_{n\in\mathbb{Z}} J^a_n z^{-n-1} = J^a(z).$$

Then we clearly have:

(ii) $Y(J^a_{-1}|0\rangle, z)|0\rangle = J^a_{-1}|0\rangle + z(\dots)$

(iii) $[T, Y(J^a_{-1}|0\rangle, z)] = \partial_z Y(J^a_{-1}|0\rangle, z)$

(v) Locality: we have in $\hat{g}_k$:

$$[J^a_n, J^b_m] = [J^a, J^b]_{n+m} + nk(J^a, J^b)\delta_{n,-m},$$

thus:

$$[J^a(z), J^b(w)] = [J^a, J^b](w)\delta(z-w) + k(J^a, J^b)\partial_w \delta(z-w).$$

It follows that $(z-w)^2[J^a(z), J^b(w)] = 0$.

Further, we have for $n<0$ in $gr V_k(g)$:

$$Y(\bar{J}^a_n|0\rangle, z) = \frac{1}{(-n-1)!}\partial_z^{-n-1}\sum_{m<0} \bar{J}^a_m z^{-m-1},$$

which motivates to define

$$Y(J^a_n|0\rangle, z) = \frac{1}{(-n-1)!}\partial_z^{-n-1} J^a(z), \text{ for } n<0.$$

Next, in commutative V.A. one has

$$Y(AB, z) = Y(A, z)Y(B, z).$$

However, the naive guess

$$\text{“}Y(J^a_{-1}J^b_{-1}|0\rangle, z) = J^a(z)\, J^b(z)\text{”}$$

is wrong, as this does not act on $V_k(\mathfrak{g})$, as:

$$J^\alpha(z) J^\beta(z) = \sum_n \Big(\sum_{i+j=-n-2} J^\alpha_i J^\beta_j \Big) z^n$$

not a field!

we need to change the order of some terms!

Def Normally ordered product of $f(z) = \sum_{n\in\mathbb{Z}} f_n z^n$, $g(z) = \sum_{n\in\mathbb{Z}} g_n z^n \in \mathrm{End}(V)[[z^{\pm 1}]]$, is

$$:f(z) g(z): = f(z)_+ g(z) + g(z) f(z)_- \,,$$ where

$$f(z)_+ = \sum_{n\geq 0} f_n z^n, \quad f(z)_- = \sum_{n<0} f_n z^n$$

We also set: $:A(z) B(z) C(z): = :A(z) :B(z) C(z)::$

Exercise: for fields $A(z), B(z)$, $:A(z) B(z):$ is a field again.

So, we set:

$$Y(J^{a_1}_{n_1} \dots J^{a_m}_{n_m} |0\rangle, z) =$$

$$= \frac{1}{(-n_1-1)!} \cdots \frac{1}{(-n_m-1)!} :\partial_z^{-n_1-1} J^{a_1}(z) \dots \partial_z^{-n_m-1} J^{a_m}(z):$$

Theorem These formulas define a V.A. structure on $V_k(\mathfrak{g})$.

$\square$ (i) $Y(|0\rangle, z) = \mathrm{id}$ by definition.

(ii) $Y(B, z)|0\rangle = B + z(\ldots)$

Induction on m (Where $B = J^{a_1}_{n_1} \ldots J^{a_m}_{n_m}|0\rangle$)

- true for $B = |0\rangle$
- Assume it's true for $Y(B,z)$. Then if $A \in g$:

$$Y(A_{-n}B, z)|0\rangle = \frac{1}{(-n-1)!} :\partial_z^{-n-1} A(z) \cdot Y(B,z): |0\rangle =$$

$$= \frac{1}{(-n-1)!}\left((\partial_z^{-n-1} A(z))_+ Y(B,z)|0\rangle + Y(B,z) \underbrace{(\partial_z^{-n-1} A(z))_- |0\rangle}_{=0} \right)$$

$$= A_{-n}B + z(\ldots).$$

(iii) Need to show:

$$[T, :A(z)B(z):] = \partial_z :A(z)B(z): .$$

Exercise: Check it, using:

$$[T, A(z)] = \partial_z A(z) \quad \text{and} \quad [T, B(z)] = \partial_z B(z).$$

(v) We checked that the fields $Y(J^a_{-1}, z)$ are mutually local for different a. Also:

Exercise: If $A(z)$, $B(z)$ are local, then $\partial_z^k A(z)$ and $\partial_z^l B(z)$ are local for $k, l \geq 0$.

Hence, the result follows from the Dong's lemma below. $\square$

Dong's lemma:

If $A(z)$, $B(z)$, $C(z)$ are mutually local, then $:A(z)B(z):$ and $C(z)$ are local.

□ We use (an easy) formula:

$$:A(w)B(w): = \mathrm{Res}_{z=0}\big(\delta(z-w)_- A(z)B(w) + \delta(z-w)_+ B(w)A(z)\big),$$

where $\delta(z-w)_- = \sum_{n<0} z^n w^{-n-1}$, $\delta(z-w)_+ = \sum_{n\geq 0} z^n w^{-n-1}$.

So, assume

$$(w-z)^s A(z)B(w) = (w-z)^s B(w)A(z)$$

$$(u-z)^s A(z)C(u) = (u-z)^s C(u)A(z)$$

$$(u-w)^s B(w)C(u) = (u-w)^s C(u)B(w),$$

and we wish to find N s.t:

$$(w-u)^N\big(\delta(z-w)_- A(z)B(w) + \delta(z-w)_+ B(w)A(z)\big)C(u)$$
$$= (w-u)^N C(u)\big(\delta(z-w)_- A(z)B(w) + \delta(z-w)_+ B(w)A(z)\big) \quad (**).$$

Take $N=3s$, then:

$$(w-u)^{3s} = (w-u)^s \sum_{r=0}^{2s} \binom{2s}{r} (w-z)^r (z-u)^{2s-r}$$

Consider the LHS of $(**)$.

· The terms with $s < r \leq 2s$ vanish

(Because of the factor $(w-z)^r$)

· The terms with $0 \leq r \leq s$:

factors $(z-u)^{2s-r}$ and $(w-u)^s$ allow to commute $C(u), A(z)$, and $C(u), B(w)$ respectively

And same in the RHS! $\square$

In fact, the following is a general tool to construct V. A's:

Th (Strong reconstruction):

Suppose V is a vector space, $|0\rangle \in V$, $T: V \to V$.

Let $a^\alpha(z) = \sum_{n \in \mathbb{Z}} a^\alpha_{(n)} z^{-n-1}$, $\alpha \in I$ (an ordered set)

such that:

(1) $[T, a^\alpha(z)] = \partial_z a^\alpha(z)$

(2) $T|0\rangle = 0$.

(3) $a^\alpha(z), a^\beta(z)$ are mutually local

(4) The vectors

$$a^{\alpha_1}_{(-j_1-1)} \cdots a^{\alpha_n}_{(-j_n-1)}|0\rangle \quad , \quad j_s \geq 0$$

span V.

Then the formula

$$Y(a^{d_1}_{(n_1)} \cdots a^{d_m}_{(n_m)} |0\rangle, z) =$$

$$= \frac{1}{(-n_1-1)!} \cdots \frac{1}{(-n_m-1)!} :\partial^{-n_1-1} a^{d_1}(z) \cdots \partial^{-n_m-1} a^{d_m}(z):$$

defines a V.A. structure on V.

The proof can be found in:
Frenkel, Ben-Zvi, "Vertex Algebras and Algebraic Curves", 4.4.1.