Vertex algebras - II

02/15/24

1. Associativity in V.A.

- Recall the locality axiom:

$$(z-w)^N [Y(A,z), Y(B,w)] = 0 \quad \text{for } N \gg 0.$$

What does it mean?

Take $A, B, v \in V$ consider

$$Y(A,z)Y(B,w)v = \sum_{n\in\mathbb{Z}} \underbrace{\Big(\sum_{m\in\mathbb{Z}} A_m z^{-m-1}\Big)(B_n v)}_{\in V((z))\ \forall n} w^{-n-1} \in V((z))((w))$$

Similarly, $Y(B,w)Y(A,z)v \in V((w))((z))$.

The spaces $V((z))((w)), V((w))((z)) \subset V[[z^{\pm 1}, w^{\pm 1}]]$ are different! $V((z))((w)) \cap V((w))((z)) = V[[z,w]][z^{-1},w^{-1}]$

intersection is taken in $V[[z^{\pm 1}, w^{\pm 1}]]$

Example. Consider $\frac{1}{z-w}$. we have:

$$\frac{1}{z-w} = \frac{1}{z(1-\frac{w}{z})} = z^{-1}\sum_{n\geq 0}\left(\frac{w}{z}\right)^n \in \mathbb{C}((z))((w))$$

$$\frac{1}{z-w} = -\frac{1}{w(1-\frac{z}{w})} = -z^{-1}\sum_{n<0}\left(\frac{w}{z}\right)^n \in \mathbb{C}((w))((z))$$

In fact:

$$(z-w)^N Y(A,z)Y(B,w)v = (z-w)^N Y(B,w)Y(A,z)v$$

in $V[[z,w]][z^{-1},w^{-1}]$

$$\Updownarrow$$

$Y(A,z)Y(B,w)v$ and $Y(B,w)Y(A,z)v$ are expansions of the same element of $V[[z,w]][z^{-1},w^{-1},(z-w)^{-1}]$ in $V((z))((w))$ and $V((w))((z))$, expanding $(z-w)^{-1}$ in either $\mathbb{C}((z))((w))$ or $\mathbb{C}((w))((z))$.)

- Motivation for associativity in V.A.'s:

Exercise 1. In a commutative V.A., we have:

$$Y(A,z)Y(B,w)C = Y(Y(A,z-w)B,w)C,$$

(follows from the associativity of the underlying algebra).

This has an analog in any V.A.

Theorem In any V.A,

$Y(A,z)Y(B,w)C$, $Y(B,w)Y(A,z)C$, and

$$Y(Y(A,z-w)B,w)C$$

are expansions of the same element of $V[[z,w]][z^{-1}, w^{-1}, (z-w)^{-1}]$

in

$$V((z))((w)), \quad V((w))((z)), \text{ and } V((w))((z-w)).$$

Before proving it, we mention a few usefull facts.

Exercise 2. $\forall A \in V$: $Y(A,z)|0\rangle = e^{zT}A$.

(Hint: Both sides lie in $V[[z]]$, satisfy the differential equation $\partial_z f = Tf$, and are equal at $z=0$.)

Exercise 3. $e^{wT} Y(A,z) e^{-wT} v = Y(A, z+w) v$

$$\cap$$
$$V((z))((w))$$

where negative powers of $(z+w)$ are expanded in positive powers of $\frac{w}{z}$.

(„exponent of the derivation is a translation operator $z \mapsto z+w$ ")

Lemma 1 (Skew Symmetry)

$$Y(A,z)B = e^{zT} Y(B,-z)A$$

$\square$

$$(z-w)^N Y(A,z)Y(B,w)|0\rangle = (z-w)^N Y(B,w)Y(A,z)|0\rangle$$

$$(z-w)^N Y(A,z)e^{wT}B = (z-w)^N Y(B,w)e^{zT}A$$

$$(z-w)^N Y(A,z)e^{wT}B = (z-w)^N e^{zT}Y(B,w-z)A$$

plug $w=0$

$$z^N Y(A,z)B = z^N e^{zT}Y(B,-z)A$$

$$Y(A,z)B = e^{zT}Y(B,-z)A \qquad \square$$

Now let's prove the theorem.

$\square$ We show that $Y(A,z)Y(B,w)C$ and $Y(Y(A,z-w)B,w)C$ are expansions in $V((z))((w))$ and $V((w))((z-w))$ of same el-t of $V[[z,w]][z^{-1},w^{-1},(z-w)^{-1}]$. Using the above lemma and exercises, we get:

$$Y(A,z)Y(B,w)C \overset{L1.}{=} Y(A,z)e^{wT}Y(C,-w)B \overset{Ex2.}{=}$$

$$= e^{wT}Y(A,z-w)Y(C,-w)B \in V((z))((w)) \qquad (1)$$

where the negative powers of $(z-w)$ are expanded in positive powers of $\frac{w}{z}$.

We also have in $V((w))((z-w))$:

$$Y(Y(A, z-w)B, w)C =$$
$$= Y(\sum_n A_{(n)}B(z-w)^{-n-1}, w)C =$$
$$= \sum_n Y(A_{(n)}B, w)C(z-w)^{-n-1} \overset{L1.}{=}$$
$$= \sum_n e^{wT} Y(C, -w) A_{(n)}B(z-w)^{-n-1}$$
$$= e^{wT} Y(C, -w) Y(A, z-w)B \qquad (2)$$

By locality of $Y(A, \cdot)$ and $Y(C, \cdot)$, (1) and (2) are expansions of the same element of $V[[z,w]][z^{-1}, w^{-1}, (z-w)^{-1}]$.
The theorem follows. $\square$

This theorem is sometimes written as
$$Y(A,z)Y(B,w) \sim \sum_{n\in\mathbb{Z}} Y(A_{(n)}B, w)(z-w)^{-n-1} \qquad \text{(OPE)}$$

it is called the <u>Operator Product Expansion</u>.

4. Corollaries of associativity.

We know that $Y(A,z)Y(B,w)C$ is an expansion in $V((z))((w))$ of an element in

$V[[z,w]][z^{-1}, w^{-1}, (z-w)^{\pm 1}]$.

Question: How to find this element?

Exercise 4. The kernel of the operator of multiplication by $(z-w)^N$ on $\mathrm{End}\, V[[z^{\pm 1}, w^{\pm 1}]]$ is spanned over $\mathbb{C}$ by $\gamma(w)\partial_w^i \delta(z-w)$, $i = 1, \dots, N-1$, $\gamma(w) \in \mathrm{End}\, V[[w^{\pm 1}]]$

(one inclusion was stated in the last lecture right after $\delta(z-w)$ was defined).

Hence, $\forall A, B \in V$, we can write:

$$[Y(A,z), Y(B,w)] = \sum_{i=0}^{N-1} \frac{1}{i!} \gamma_i(w) \partial_w^i \delta(z-w),$$

and one can check that $\gamma_i(w)$ are fields, as $Y(A,z)$, $Y(B,w)$ are fields.

Recall: If $\varphi(z) = \sum \varphi_i z^i$, $\psi(z) = \sum \psi_i z^i$ - fields,

$:\varphi(z)\psi(w): = \varphi_+(z)\psi(w) + \psi(w)\varphi_-(z)$,

where $\varphi(z)_+ = \sum_{n \geq 0} \varphi_i z^i$, $\varphi(z)_- = \sum_{i<0} \varphi_i z^i$.

Lemma 2. For fields $\varphi(z), \psi(z)$ TFAE:

(1) $[\varphi(z), \psi(w)] = \sum_{i=0}^{N-1} \frac{1}{i!} \gamma_i(w) \partial_w^i \delta(z-w)$

(2) $\varphi(z)\psi(w) = \sum_{i=0}^{N-1} \gamma_i(w) \left(\frac{1}{(z-w)^{i+1}} \right)_{|z|>|w|} + :\varphi(z)\psi(w):$

and

$$\psi(w)\varphi(z) = \sum_{i=0}^{N-1} \gamma_i(w) \left(\frac{1}{(z-w)^{i+1}} \right)_{|w|>|z|} + :\varphi(z)\psi(w):,$$

where $\left(\frac{1}{(z-w)^{i+1}} \right)_{|z|>|w|}$ means that we expand this in $\mathbb{C}((z))((w))$.

Exercise 5. Prove it.

$Y(A,z), Y(B,w)$ satisfy (1) $\Rightarrow$ satisfy (2).

It follows that $\forall A, B, C \in V$, the series $Y(A,z)Y(B,w)C \in V((z))((w))$ is an expansion of:

$$\Big(\underbrace{\sum_{i=0}^{N-1} \frac{\gamma_i(w)}{(z-w)^{i+1}}}_{\in\, V((w))[(z-w)^{-1}]} + \underbrace{:Y(A,z)Y(B,w):}_{\in\, V[[z,w]][z^{-1},w^{-1}]}\Big)C$$

Using the Taylor formula, the expansion of this element in $V((w))((z-w))$ is:

$$\sum_{i=0}^{N-1} \frac{\gamma_i(w)}{(z-w)^{i+1}} + \sum_{m\geq 0} \frac{(z-w)^m}{m!} :\partial_w^m Y(A,w)\cdot Y(B,w):)C$$

Now, compare the coefficients of $(z-w)^k$ in this formula and in the formula (OPE).

- For $k\geq 0$ we get:

for $n<0$:

$$Y(A_{(n)}B,z) = \frac{1}{(-n-1)!} :(\partial_z^{-n-1} Y(A,z))\cdot Y(B,z): \quad (*)$$

Recall that in the last lecture, just after defining a V.A., we mentioned that in any V.A. $TA = A_{(-2)}|0\rangle$

Thus, setting $n=-2$, $B=|0\rangle$ in $(*)$, we get:

$$Y(TA,z) = \partial_z Y(A,z).$$

Iterating $(*)$ we obtain:

<u>Proposition</u>: $\forall\ A^1,\dots,A^m \in V$, $n_1,\dots,n_m<0$:

$$Y(A^1_{(n_1)}\dots A^m_{(n_m)}|0\rangle, z) =$$

$$= \frac{1}{(-n_1-1)!} \cdots \frac{1}{(-n_m-1)!} :\partial_z^{-n_1-1} Y(A^1, z), \dots, \partial_z^{-n_m-1} Y(A^m, z):$$

This tells that after we set $Y(J^a_{-1}, z) = J^a(z)$ in $V_k(\mathfrak{g})$, we did not have choice for what $Y(J^{a_1}_{n_1} \cdots J^{a_m}_{n_m}, z)$ should be!

- Now compare the coefficients of $(z-w)^k$ for $k<0$. We get:

$$\gamma_i(w) = Y(A_{(i)}B, w).$$

So we get:

$$Y(A, z)Y(B, w) = \sum_{n\geq 0} \frac{Y(A_{(n)}B, w)}{(z-w)^{n+1}} + :Y(A, z)Y(B, w):;$$

and also:

$$[Y(A, z), Y(B, w)] = \sum_{n\geq 0} \frac{1}{n!} Y(A_{(n)}B, w)\partial_w^n \delta(z-w),$$

which, after expanding the terms boils down to:

$$[A_{(m)}, B_{(k)}] = \sum_{n\geq 0} \binom{m}{n} (A_{(n)}B)_{(m+k-n)}.$$

These formulas will be important later!