Invariant theory 1, 01/13/25

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1) Motivation from Linear algebra

A typical basic problem in the subject is: given a finite dimensional $\mathbb{C}$-vector space, U, form a related $\mathbb{C}$-vector space, V e.g. (a) linear operators $U \to U$ or

(b) bilinear forms $U \times U \to \mathbb{C}$ (maybe w. a symmetry or skew-symmetry condition)

(c) or a more general tensors.

Choosing a basis in U allows us to write elements of V as collections of scalars that can often (say in (a), (b)) be arranged into a matrix (or a generalization of such in (c)). When we change the basis of U the matrix (or its generalization) changes and we want to find "canonical forms" or more generally to find necessary and/or sufficient conditions for one matrix to be obtained from another via a change of basis. The Jordan normal form theorem (addressing (a)) is an example of such result. A classification of homogeneous deg d polynomials in n variables (up to a linear change of variables) is another – this problem was of great importance in the 19th century. It only has nice solutions for small n and/or d.

Of course, we can phrase these problems in a more educated way. The group $G := GL(U)$ of invertible linear operators $U \to U$ acts on V in a natural way, e.g. for $V = \operatorname{End}(U)$ we have the conjugation action, $g.A = gAg^{-1}$ ($g \in GL(U), A \in V$). We want to classify the orbits (i.e. find a bijection between the set of orbits and a known set)

We can also consider more general groups: $G = SL(U)$ (in Linear algebra this corresponds to the situation when U comes w. fixed volume form), $G = O(U)$ (for an orthogonal space U i.e. a space equipped w. a nondegenerate symmetric bilinear form), $G = Sp(U)$ (for a symplectic space) or product of such groups (e.g. $G = GL(U_1) \times GL(U_2)$ acting on $V = \operatorname{Hom}(U_1, U_2)$ by $(g_1, g_2).A = g_1 A g_2^{-1}$).

So our motivating question is:

Question: how to classify orbits for a "nice" action of a nice group G on a vector space V by linear transformations.

2) Invariants

Let $\mathrm{Fun}(V)$ denote the set of functions $V \to \mathbb{C}$, it is naturally a (commutative associative unital) $\mathbb{C}$-algebra. Note that the action of G on V gives rise to a G-action on $\mathrm{Fun}(V)$, $[g.f](v) := f(g^{-1}v)$, by algebra automorphisms.

Definition: An element $f \in \mathrm{Fun}(V)$ is called G-invariant if $g.f = f$ $\forall g \in G \Leftrightarrow f$ is constant on G-orbits. The subset of all G-invariants is denoted by $\mathrm{Fun}(V)^G$, this is a subalgebra.

Let $\sim_G$ denote the equivalence relation on V given by $v_1 \sim_G v_2$ if $v_1 \in G v_2$. A basic observation is:

$$v_1 \sim_G v_2 \Leftrightarrow f(v_1) = f(v_2) \ \forall f \in \mathrm{Fun}(V). \qquad (1)$$

A problem here is that we don't really have tools to deal with <u>all</u> functions: we need to restrict to something manageable. In this course we will deal with <u>polynomial</u> functions (=polynomials in coordinates w.r.t. a basis). Let $\mathbb{C}[V] \subset \mathrm{Fun}(V)$ denote the subalgebra of polynomials. It's G-stable (G acts by linear changes of variables) so we can consider the subalgebra of invariants, $\mathbb{C}[V]^G$. This is one of the main objects of study in this course.

An advantage of $\mathbb{C}[V]^G$ over $\mathrm{Fun}(V)^G$ is that one can actually study the former. A disadvantage is that one cannot fully extract the information about the orbits from knowing $\mathbb{C}[V]^G \subset \mathbb{C}[V]$. Define equivalence relation $\sim_{inv}$ on V by: $v_1 \sim_{inv} v_2 \Leftrightarrow f(v_1)=f(v_2)\ \forall f\in \mathbb{C}[V]^G$.

Example 0: Let $G=\mathbb{C}^\times$, the multiplicative group & V be any finite dimensional space w. $\mathbb{C}^\times \curvearrowright V$ by dilations: $z.v=zv$, $z\in\mathbb{C}^\times$, $v\in V$. Choosing a basis in V we identify $\mathbb{C}[V]$ w. $\mathbb{C}[x_1,\dots x_n]$. The action of $\mathbb{C}^\times$ on $\mathbb{C}[x_1,\dots x_n]$ is as follows: $z\in\mathbb{C}^\times$ multiplies x_i by z^{-1}, and so a deg d monomial by z^{-d}. From here we see that $\mathbb{C}[V]^G=\mathbb{C}$, the scalars. So (1) (badly) fails if we restrict to polynomial invariants: $\sim_{inv}$ is trivial.

Example 1: Let $G=GL(U)$ acting on $V=\mathrm{End}(U)$ by conjugations. Define $X_1,\dots X_n\in\mathbb{C}[V]$ as the coefficients of the characteristic polynomial: $\det(\lambda\cdot Id-A)=\lambda^n+\sum_{i=1}^n(-1)^i X_i(A)\lambda^{n-i}$. Then $X_1,\dots,X_n\in\mathbb{C}[V]^G$. One can show (we will cover a more general result) that $\mathbb{C}[V]^G$ is the free algebra in generators $X_1,\dots X_n$. (1) still fails but not as badly: $A_1\sim_{inv}A_2$ iff A_1, A_2

have the same char. polynomials (but they may have different Jordan normal forms).

3) Work of Hilbert

So, while computing the invariants doesn't allow to fully classify the G-orbits in V, it still gives a lot of information. The most basic question in the study of $\mathbb{C}[V]^G$ is

Question 1: Is $\mathbb{C}[V]^G$ finitely generated?

Constructing finite sets of generators in various cases was an important part of the 19th century Mathematics. Then came Hilbert. He wrote two papers, in 1890 & 1893 that completely revolutionized the subject (so that it didn't recover until the 2nd half of the 20th century) and also laying foundations of modern Commutative algebra.

In modern language the main theorem in Hilbert's 1890 paper is as follows

Thm 1: Let G be a reductive algebraic group acting on V via a rational representation. Then $\mathbb{C}[V]^G$ is finitely generated.

For example, all groups mentioned in Sec 1 are reductive & all representations mentioned there are rational.

Rem: A philosophical implication of the theorem is that to check $v_1 \sim_{inv} v_2$ one needs to do only finitely many checks: if $f_1, \ldots f_k$ generate $\mathbb{C}[V]^G$, then we need to check $f_i(v_1) = f_i(v_2)$ $\forall$ $i = 1, \ldots k$.

Here's a geometric implication of Thm 1. The algebra $\mathbb{C}[V]^G$ has no nilpotents so we can form the corresponding variety, to be denoted by $V/\!\!/G$. The inclusion $\mathbb{C}[V]^G \hookrightarrow \mathbb{C}[V]$ gives rise to a dominant morphism $\pi: V \to V/\!\!/G$. The pair $(V/\!\!/G, \pi)$ (or, abusing the terminology, $V/\!\!/G$ itself) is called the categorical quotient (for the action of G on V).

Note that π is G-invariant (exercise) so every fiber is a union of orbits. Using techniques similar to the proof of Thm 1 one can show – we will do this later:

Proposition: 1) π is surjective.

2) Every fiber of π contains a unique closed (in Zariski topology) G-orbit.

So $V/\!/G$ parameterizes the closed G-orbits in V.

The 2nd paper of Hilbert on the subject, from 1893, addressed (in our terminology) a description of $\sim_{inv}$.

Exercise: Use 2) of Proposition to show that the Zariski closure of every G-orbit in V contains a unique closed orbit.

Hilbert in what is known now as the Hilbert-Mumford theorem gave an efficient way of determining that closed orbit. We postpone stating the general result but give a classical example.

Example: Let $G = SL_2$, $V = S^n(\mathbb{C}^2)$ a.k.a. the space of homogeneous deg n polynomials in 2 variables, x, y. We want to determine the inv-equivalence class of 0 = the common set of zeroes of all elements of $\mathbb{C}[V]^G$ that vanish at 0. Note that every $f \in V$ uniquely (up to scalar factors) decomposes into the product of linear factors (exercise; hint: reduce this to the usual polynomials in $z := y/x$). Hilbert proved that $f \sim_{inv} 0$ $(f \in V) \Leftrightarrow f$ has a linear factor w.

multiplicity $> n/2$. Note that one doesn't know how to write generators of $\mathbb{C}[V]^G$ for n sufficiently large.

Rem: The two Hilbert papers are best known for their auxiliary results: the basis theorem in the 1890 paper & Nullstellensatz in the 1893 paper.

4*) Applications to Algebraic geometry

After revolutionary work of Hilbert, Invariant theory lost its prominent status. It gained prominence again after David Mumford used a geometric version of the theory to construct moduli spaces in Algebraic geometry.

In the previous sections we talked about linear actions $G \curvearrowright V$. Both Theorem and Proposition in Sec 3 continue to hold for the actions of G on affine varieties (or finite type affine schemes) X. In particular, we still have the categorical quotient $X/\!/G$ whose points parameterize the closed G-orbits.

What if we want to parameterize some other orbits? For example, by its very definition, the projective space $\mathbb{P}(V)$ parameterizes the nonzero $\mathbb{C}^\times$-orbits in V for the dilation action.

For more general $G \curvearrowright X$ we could try to immitate how

$\mathbb{P}(V)$ is defined and:

1) Cover some open G-stable subset $X^\circ \subset X$ w. G-stable open affines: $X^\circ = \bigcup_{i=1}^{k} X_i$

2) Form quotients $X_i /\!/ G$ and glue them along their "intersections" $(X_i \cap X_j)/\!/G$.

However many things can go wrong with that procedure especially if we want our result to be a vanilla variety (or separated scheme). It turns out that we haven't used all of the structure in our basic example. Namely, to a choice of character, χ, of G = an algebraic group homomorphism $G \to \mathbb{C}^\times$, one can assign an open subset of "χ-semistable" points $X^{\chi\text{-}ss} \subset X$ and its cover by open G-stable affines so that 1) & 2) give a scheme, $X/\!/^{\chi} G$, a GIT quotient parameterizing the closed G-orbits in $X^{\chi\text{-}ss}$. In our basic example, we take $\chi = \mathrm{id}$ and get $X^{\chi\text{-}ss} = V \setminus \{0\}$ & $X/\!/^{\chi} G = \mathbb{P}(V)$.

The usefulness of this construction, as realized by Mumford is that one can use it to produce the "coarse moduli spaces" of various objects in Algebraic geometry – i.e. varieties / schemes parameterizing these objects.