Invariant theory 10, 2/12/25

1) Example of $SL_3 \curvearrowright S^3(\mathbb{C}^3)$, finished
2) Sections

Refs: [PV], Secs 8.1, 8.8.

1) Example of $SL_3 \curvearrowright S^3(\mathbb{C}^3)$, finished

Here's where our study of θ-groups is. Let $G_0 \curvearrowright \mathfrak{g}_1$ be as before. Let $\mathfrak{a} \subset \mathfrak{g}_1$ be a Cartan subspace and $W_\theta \subset GL(\mathfrak{a})$ be the Weyl group. In Lec 7 we have proved the Chevalley restriction theorem: $\mathbb{C}[\mathfrak{g}_1]^{G_0} \xrightarrow{\sim} \mathbb{C}[\mathfrak{a}]^{W_\theta}$ and in Lec 8 we proved that at least when G_0 is semisimple both sides are the isomorphic to the algebra of polynomials in $\dim \mathfrak{a}$ variables.

In Lec 9 we've learned to construct some examples of G_0 & $\mathfrak{g}_1$. It turns out that most interesting examples, ones that cannot be handled using usual "Linear algebra," arise when $\mathfrak{g}$ is exceptional or when $\mathfrak{g} = \mathfrak{so}_8$ & $\theta \in Aut(\mathfrak{g})$ projects to an order 3 element in $Aut(\mathfrak{g})/Aut(\mathfrak{g})^\circ$.

After we learned to construct $G_0 \curvearrowright \mathfrak{g}_1$, a natural task is to compute $\mathfrak{a}$ & W_θ. Take $\mathfrak{g} = \mathfrak{so}_8$ and we consider the order 3 automorphism θ of $\mathfrak{g}$ constructed in Sec 1.3 of Lec 9 so that $\mathfrak{g}_0 = \mathfrak{sl}_3$ & $\mathfrak{g}_1 = S^3(\mathbb{C}^3)$. Recall that in Sec 2 of Lec 9 we constructed a

Cartan subalgebra $\mathfrak{h}' \subset \mathfrak{g}$ preserved by θ where the ε-eigenspace, $\mathfrak{a}$, is 2-dimensional. We've seen that $\mathfrak{a}$ is a Cartan subspace of $\mathfrak{g}_1$. We've also seen that $\mathfrak{h}' = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{a})$.

1.1) Computation of W_θ.

Let T' denote the maximal torus in G_{sc} w. $Lie(T') = \mathfrak{h}'$. Consider the group $W = N_G(\mathfrak{h}')/T'$ of $\mathfrak{g}$ w.r.t. $\mathfrak{h}'$. We write θ' for $\theta|_{\mathfrak{h}'}$. Then $\theta' W \theta'^{-1} = W$. Let $W' = \{w \in W \mid \theta' w = w\theta'\}$

Proposition (Vinberg)

1) W' preserves $\mathfrak{a} \subset \mathfrak{h}'$ and acts on it faithfully. The image of W' in $GL(\mathfrak{a})$ is W_θ.

2) W_θ is a complex reflection group G_4 (to be described below).

The proof of 1) (in a more general setting) is in [V], Sec. 8. Then 2) is a result of a computation.

Now we define the group G_4. Let Γ be the Kleinian group of type E_6. Recall that it is constructed as follows. Let $\underline{\Gamma} \subset SO_3(\mathbb{R})$ be the group of rotations of a regular tetrahedron, it is isomorphic to the alternating group $\mathfrak{A}_4$ (w. 12 elements). Then Γ is the preimage of $\underline{\Gamma}$ under $SU_2 \twoheadrightarrow SO_3(\mathbb{R})$, so has

24 elements.

In particular, the inclusion of Γ into $SU_2 (\subset SL_2(\mathbb{C}))$ gives rise to a 2-dimensional, in fact, irreducible representation, U_1, of Γ. It is self-dual b/c it has an invariant symplectic form.

Also, $\underline{\Gamma}$ (and hence Γ) has two nontrivial 1-dimensional representations to be denoted by $\mathbb{C}_\varepsilon, \mathbb{C}_{\varepsilon^2}$, they are dual to each other. Set $U_\varepsilon := U_1 \otimes \mathbb{C}_\varepsilon$.

Fact/extended exercise.

1) Γ acting on U_ε is a complex reflection group (hint: describe the 7 conjugacy classes in Γ & compute the character of U_ε)
2) Let d_1, d_2 be the degrees of free homogeneous generators of $\mathbb{C}[U_\varepsilon]^\Gamma$. Then $d_1 = 4, d_2 = 6$.

Hints: Show that $d_i \geq 3$ by elementary means. Then apply the Chevalley-Shephard-Todd thm (Sec. 2.1 of Lec 6) to show $d_1 d_2 = 24$.

Remark: It is classically known (Poincare?) that $\mathbb{C}[\mathfrak{g}_1]^{G_0} = \mathbb{C}[S,T]$ for polynomials of degrees 4 (for S) and 6 (for T). However their construction and the proof of the equality above are not immediate. See Sec. 0.14 in [PV] for details.

2) Sections

We've seen that θ-groups have polynomial algebras of invariants (at least when G_0 is semisimple) and finitely many orbits in each fiber of the quotient morphism. In Remark in Sec 1 of Lec 5 we have mentioned two more favorable properties:

(i) All fibers of π have the same dimension.

(ii) π has a section.

Below we will examine these properties. Let G be a connected reductive group acting on its rational representation V. We assume:

(a) $\mathbb{C}[V]^G$ is a polynomial algebra

(b) Each fiber of $\pi: V \to V/\!\!/G$ consists of finitely many orbits.

2.1) Flatness.

Proposition: If (a) & (b) hold, then π is flat.

Sketch of proof: We have $\forall X \in V/\!\!/G \Rightarrow$ all components of $\pi^{-1}(X)$ have $\dim \geqslant \dim V - \dim V/\!\!/G$.

On the other hand, the dimension of orbits is upper semi-continuous: $\{v \in V \mid \dim Gv \geqslant d\}$ is Zariski open. Combining these with (b) we see that all fibers of π have the same dimension.

Then we use the following commutative algebra fact:

a morphism from a Cohen-Macaulay (e.g. smooth) variety to a smooth variety is flat iff all fibers have the same dimension, see [E], Sec. 18.4 $\square$

2.2) Existence & construction.

We have the following result due to Knop.

Theorem: Suppose G is semisimple and (a), (b) hold. Then $\pi: V \to V/\!\!/G$ admits a section $\iota: V/\!\!/G \hookrightarrow V$.

We'll explain ideas of a proof in Sec 2.4 & give a proof in a bonus note. We are going to look for $S := im(\iota)$ of special form.

Pick $e \in \pi^{-1}(0)$ w. orbit of maximal dimension so that

$$\dim Ge = \dim V - \dim V/\!\!/G.$$

We'll find an affine subspace $S \subset V$ w. $e \in S$ s.t. S is transverse to Ge & is stable under a suitable action of $\mathbb{C}^\times$ fixing e.

It turns out that these properties imply $\pi|_S : S \xrightarrow{\sim} V/\!\!/G$.

Here's how the action of $\mathbb{C}^\times$ is constructed. Consider the action of $\mathbb{C}^\times$ on V by dilations. It preserves $\pi^{-1}(0)$, hence every irreducible component of $\pi^{-1}(0)$ including $\overline{Ge}$. Hence it preserves Ge as the open G-orbit in $\overline{Ge}$. Let

$$Z := \mathrm{Stab}_G(e) \ \& \ \tilde{Z} := \mathrm{Stab}_{G\times\mathbb{C}^\times}(e)$$

Exercise: $Z \triangleleft \tilde{Z}$ & $\tilde{Z}/Z \hookrightarrow G\times\mathbb{C}^\times/G \xrightarrow{\sim} \mathbb{C}^\times$.

Lemma: Let p denote the projection $\tilde{Z} \twoheadrightarrow \mathbb{C}^\times$. Then $\exists$ homom. $\iota: \mathbb{C}^\times \to \tilde{Z}$ & $k>0$ w. $p\circ\iota(t) = t^k$ $\forall\, t\in\mathbb{C}^\times$.

Sketch of proof:

Set $F := \tilde{Z}/R_u(\tilde{Z})$ so that p factors through F; note that F is reductive. By Levy's Thm (Sec 6.4 in [OV]) $\tilde{Z} \to F$ admits a section $F \hookrightarrow \tilde{Z}$ so it's enough to construct $\iota: \mathbb{C}^\times \hookrightarrow F$ w. $p\circ\iota(t) = t^k$. Then one observes that $F^\circ \twoheadrightarrow \mathbb{C}^\times$. As any connected reductive group, F° decomposes into product $Z(F^\circ)^\circ (F^\circ, F^\circ)$ w. finite intersection & since (F°, F°) is s/simple it maps trivially to $\mathbb{C}^\times$. So we need to construct $\iota: \mathbb{C}^\times \hookrightarrow Z(F^\circ)^\circ$ w. $p(\iota(t)) = t^k$. The existence of such ι is left as an exercise. $\square$

Consider the action of $\mathbb{C}^\times$ on V via ι. It fixes e & normalizes the action of G (as $\tilde{Z} \subset G\times\mathbb{C}^\times$). So it fixes $\mathfrak{g}e \subset V$. Let S_0 be a $\mathbb{C}^\times$-stable complement to $T_e Ge = \mathfrak{g}e$ in V. Set $S := e + S_0$. It's $\mathbb{C}^\times$-stable & transverse to Ge by the construction.

We'll show later that $\pi|_S : S \xrightarrow{\sim} V/\!/G$ proving Theorem.

2.3) Examples & motivations.

2.3.1) Adjoint action & Slodowy slices

Let G be a connected reductive group & $\mathfrak{g} = \mathrm{Lie}(G)$. We are concerned with the adjoint action of G on $\mathfrak{g}$.

A basic tool to study nilpotent orbits in $\mathfrak{g}$ is the following result:

Thm (Jacobson-Morozov): $\forall$ nilpotent element $e \in \mathfrak{g}$ $\exists\, h, f \in \mathfrak{g}$ s.t. the defining relations of $\mathfrak{sl}_2$ are satisfied:

$$[h,e] = 2e, \ [h,f] = -2f, \ [e,f] = h.$$

For two different proofs see [CM], Sec. 3.3 & Exercises 16-19 to Sec. 4.1 in [OV]. The triple (e,h,f) is called an $\mathfrak{sl}_2$-triple.

In particular, we can construct a transverse slice S to Ge known as the Slodowy slice in this generality: note that $\ker(\mathrm{ad} f) \oplus \mathrm{im}(\mathrm{ad}\, e) = \mathfrak{g}$ (& of course $\mathfrak{g}.e = \mathrm{im}(\mathrm{ad}\, e)$). A $\mathbb{C}^\times$-action is constructed as follows: the elements e, h, f give rise to a homomorphism $\mathfrak{sl}_2 \to \mathfrak{g}$ which integrates to $SL_2 \to G$. Composing this $\mathbb{C}^\times \to SL_2$, $t \mapsto \mathrm{diag}(t, t^{-1})$, we get a homomorphism $\gamma: \mathbb{C}^\times \to G$. We have $\gamma(t).e = t^2 e$. Then we take the action given by $t.x = \gamma(t)^{-1} t^2 e$. Clearly $\ker(\mathrm{ad} f)$ is stable under this action.

We set $S := e + \ker(\mathrm{ad} f)$.

2.3.2) Kostant slice

A special case of this construction was discovered previously by Kostant in [Ko]. Let e_i, h_i, f_i, $i=1,\dots,r$, be the Chevalley generators of $\mathfrak{g}$. Set $e := \sum_{i=1}^{r} e_i$, $h = 2\rho^\vee$, where $\rho^\vee = \sum_{i=1}^{r} \omega_i^\vee$. To define f let $n_i \in \mathbb{Z}$ be defined by $h = \sum_{i=1}^{r} n_i h_i$. Set $f = \sum_{i=1}^{r} n_i f_i$.

Important exercise: 1) Show that (e,h,f) is an $\mathfrak{sl}_2$-triple.

2) Show that $\dim Ge = \dim \mathfrak{g} - r$ & hence Ge is open in $\pi^{-1}(0)$.

Hint: all irreducible summands of the representation of $\mathfrak{sl}_2$ in $\mathfrak{g}$ coming from the $\mathfrak{sl}_2$-triple (e,h,f) are odd-dimensional hence

$$\dim \ker(\operatorname{ad} h) = \dim \ker(\operatorname{ad} e).$$

So in this case $S \xrightarrow{\sim} \mathfrak{g}/\!/G$, proved by Kostant.

2.3.3) θ-groups.

Now suppose θ is a finite order automorphism of G. Consider the action of G_0 on $\mathfrak{g}_1$. Let $e \in \mathfrak{g}_1$ be a nilpotent. An (up) graded version of the Jacobson-Morozov theorem says that one can find $h \in \mathfrak{g}_0$, $f \in \mathfrak{g}_{-1}$. Both proofs mentioned above can be adapted to the graded setting. It is easy to see that $(\ker(\operatorname{ad} f) \cap \mathfrak{g}_1) \oplus \mathfrak{g}_0 . e = \mathfrak{g}_1$ (exercise). Applying this to the case when $G_0 e$ is open in $\pi^{-1}(0)$

we see that $e+(\ker(\operatorname{ad} f)\cap \mathfrak{g}_1) \xrightarrow{\sim} \mathfrak{g}_1/\!/G_0$ (for s/simple G_0).

2.3.4) $SL_3 \curvearrowright S^3(\mathbb{C}^3)$

This is a special case of Sec 2.3.3 but also the most classical of the cases where a section is known. Namely, let x,y,z be a basis. Then consider

$$S := \{x^3+y^2z+pxz^2+qz^3 \mid p,q\in\mathbb{C}\}$$

known as the Weierstrass section

Exercise: 1) Show that $e=x^3+y^2z$ is nilpotent by observing that $\operatorname{diag}(t,t^4,t^{-5}).e = t^3e \quad \forall\, t\in\mathbb{C}^\times$

2) Show that $\mathfrak{sl}_3.e$ is the span of all monomials but xz^2 & z^3.

3) Show that S is $\mathbb{C}^\times$-stable for a suitable $\mathbb{C}^\times$-action & transverse to $SL_3.e$. So $S \xrightarrow{\sim} S^3(\mathbb{C}^3)/\!/SL_3$.

Remark: Kostant slice is very important for various aspects of Geometric Representation theory. One example: derived Satake of Bezrukavnikov-Finkelberg. Generalizations of this from relative geometric Langlands likely require a more general setting of the theorem.

2.4) Steps to prove the theorem

Step 1: Using that the fibers of $\pi: V \to V/\!\!/G$ are equi-dimensional prove that $\operatorname{codim}_V \{v \in V \mid d_v\pi \text{ is surjective}\} \geqslant 2$. Techniques involved are similar to those of Steps 1&2 of the proof of Prop 1 in Lec 8.

Step 2: From the construction of the $\mathbb{C}^\times$-action $\exists\, k \in \mathbb{Z}_{>0}$ & $\gamma: \mathbb{C}^\times \to G$ s.t. the $\mathbb{C}^\times$-action fixing S & e is given by $t.v = t^k \gamma(t) v$. So if we consider the action of $\mathbb{C}^\times$ on $V/\!\!/G$ induced by $(t,v) \mapsto t^k v$, we see that $S \to V/\!\!/G$ is $\mathbb{C}^\times$-equivariant. This together with the transversality of the intersection $S \cap Ge$ can be used to show:

(a) The action of $\mathbb{C}^\times$ on S contracts it to e.

(b) $\pi_S^{-1}(0) = \{e\}$

(c) $\forall s \in S \Rightarrow T_s S \oplus T_s Gs = V$

Step 3: From (b) and the claim that the $\mathbb{C}^\times$-action is contracting one deduces that $\pi|_S$ is finite. Note that S & $V/\!\!/G$ are isomorphic affine spaces. For a finite endomorphism of an affine space the locus where it is ramified is a divisor. On the other hand (c) implies that the map $G \times S \to V, (g,s) \mapsto gs$ is smooth. From here & Step 1 one deduces that $\mathfrak{M}_S : S \to V/\!\!/G$ is unramified away from codim 1. Hence it is etale. A finite etale endomorphism of an affine space is an automorphism.