Bonus: Proof of Thm from Sec 2.2 in Lec 10.

Our goal here is to prove a stronger version of Theorem in Sec. 2.2 of Lec 10. Namely, we consider an action of a (connected) s/simple group G on a vector space V s.t.

(a) $V/\!/G$ is an affine space

(b) Every fiber of $\pi: V \to V/\!/G$ contains finitely many orbits

We've seen in Sec 2.1 of Lec 21 that π is flat.

We pick $e \in \pi^{-1}(0)$ s.t. Ge is open in $\pi^{-1}(0)$. In Sec 2.2 of Lec 10 we have produced a homomorphism $\mathbb{C}^\times \to G \times \mathbb{C}^\times$ of the form $t \mapsto (\gamma(t), t^k)$ s.t. $t.e = e$ for the resulting $\mathbb{C}^\times$-action on on V. Then we take a $\mathbb{C}^\times$-stable complement S_0 to $\mathfrak{g}.e$ in V and set $S := e + S_0$. This is a $\mathbb{C}^\times$-stable affine subspace intersecting Ge at e transversally.

Theorem (Knop): $\pi: S \xrightarrow{\sim} V/\!/G$.

Rem: one can relax (b) to $\overline{Ge}$ being an irreducible component of $\pi^{-1}(\pi(0))$ and remove (a) altogether (premium exercise). A more interesting question is how to relax to semisimplicity of G.

We are now going to implement the strategy of the proof described in Sec 2.4 of Lec 10.

1) Step 1: locus of smooth points of π

Here we are proving the following:

Claim: $V' := \{v \in V \mid d_v\pi \text{ is not surjective}\}$ has codim $\geqslant 2$ in V.

We will use the following easy fact

Fact: Let X, Y be irreducible varieties & $\pi: X \to Y$ be a dominant morphism. Then

$$\{x \in X^{reg} \mid \pi(x) \in Y^{reg} \ \& \ d_x\pi \text{ is surjective}\}$$

is Zariski open & non-empty.

Apply this to $\pi: V \to V/\!/G$ (both varieties are smooth). Since π is G-invariant, V' is G-stable.

Assume the contrary of Claim and take an irreducible component $D \subset V'$ of codim 1. Take an irreducible polynomial $f \in \mathbb{C}[V]$ defining D. Step 1 of the proof of Proposition 1 in Lec 8 shows $f \in \mathbb{C}[V]^G$. Step 2 of that proof shows that $\pi(D) \simeq D/\!/G$ is also a divisor defined by f but in $V/\!/G$. Note that this description implies $D = \pi^{-1}(\pi(D))$ as <u>subschemes</u> of V.

We are going to show that $\exists\ x \in D^{reg}$ s.t. $d_x\pi$ is surjective leading to a contradiction. First notice that since f is irreducible, $D^1 := \{x \in D^{reg} \mid d_x f \neq 0\}$ is non-empty and open. By Fact applied to $\pi : D \longrightarrow D/\!/G$ we see that

$$D^2 := \{x \in D^{reg} \mid \pi(x) \in (D/\!/G)^{reg},\ d_x(\pi|_D) \text{ is surjective}\}$$

is open & non-empty. We claim that $d_x\pi$ is surjective $\forall$ $x \in D^1 \cap D^2$. This follows from the next commutative diagram of SES's:

$$\begin{tikzcd}
0 \arrow[r] & T_x D \arrow[r] \arrow[d, "d_x(\pi|_D)"'] \arrow[dr, phantom, "\text{inclusions}"] & T_x V \arrow[r, "d_x f"] \arrow[d, "d_x\pi"] & \mathbb{C} \arrow[r] \arrow[d, "id"] & 0 \\
0 \arrow[r] & T_{\pi(x)}(D/\!/G) \arrow[r] & T_{\pi(x)}(V/\!/G) \arrow[r, "d_{\pi(x)} f"'] & \mathbb{C} \arrow[r] & 0
\end{tikzcd}$$

We arrive at a contradiction w. choice of D.

2) Step 2: contracting $\mathbb{C}^\times$-action & consequences

Note that the action of $\mathbb{C}^\times$ on S is linear (via $S \xrightarrow{\sim} S_0$) therefore in a suitable basis it looks like $t(u_1, \dots u_r) = (t^{n_1}u_1, \dots t^{n_r}u_r)$. We say that the action is contracting if all $n_i > 0$.

Lemma: The $\mathbb{C}^\times$-action on S is contracting.

Proof: Assume the contrary: $\exists\, u \in S_0 \setminus \{0\}$ w. $t.u = t^{-\ell} u$ for $\ell \geq 0$.

Step 1: Since $t.v = t^k \gamma(t) v$ $\forall v \in V$ & $\gamma(t) \in G$, the morphism π intertwines this action of $\mathbb{C}^\times$ on V with the action of $\mathbb{C}^\times$ on $V/\!/G$ induced by $(t,v) \mapsto t^k v$, which is contracting.

Now consider $v = e + u$. We claim that $\pi(v) = 0$

We have $t.v = e + t^{-\ell} u$. It follows that $\lim_{t\to\infty} t.v$ exists in V and equals e. So

(1) $\quad \lim_{t\to\infty} t.\pi(v) = \lim_{t\to\infty} \pi(t.v) = \pi(\lim_{t\to\infty} t.v) = \pi(e) = 0$

Since the action of $\mathbb{C}^\times$ on $V/\!/G$ is contracting, (1) implies $\pi(v) = 0$ (where we abuse the notation & write 0 for $\pi(0)$).

Step 2: We can replace u in Step 1 with au $\forall a \in \mathbb{C}^\times \setminus \{0\}$. It follows that $\pi(e + au) = 0$ $\forall a \in \mathbb{C}$. On the other hand, S intersects Ge transversally at e. Since Ge is open in $\pi^{-1}(0)$, we see that e is an isolated point of $S \cap \pi^{-1}(0)$. This contradicts $e + au \in S \cap \pi^{-1}(0)$ and finishes the proof $\square$

We are going to deduce two corollaries from the lemma.

Corollary 1: $(\pi|_S)^{-1}(0) = \{e\}$ (as a subset)

Proof:

Note that since π is $\mathbb{C}^\times$-equivariant, $(\pi|_S)^{-1}(0) = \pi^{-1}(0) \cap S$ is $\mathbb{C}^\times$-stable. As was mentioned in Step 2 of the proof of Lemma, e is an isolated point of $(\pi_S)^{-1}(0)$. Since the $\mathbb{C}^\times$-action on S is contracting, this implies $(\pi|_S)^{-1}(0) = \{e\}$. $\square$

Corollary 2: $T_s S \oplus T_s Gs = V$ $\forall s \in S$.

Proof: First we observe that the set $\{s \in S \mid T_s S \oplus T_s Gs\}$ is $\mathbb{C}^\times$-stable and contains e. It remains to show that this set is Zariski open. First, observe that since the action of $\mathbb{C}^\times$ normalizes G and contracts S to e, we have $\dim Gs \geq \dim Ge$ $\forall s \in S$. On the other hand, Ge already has the maximal possible dimension for an orbit in V. So $\dim Gs = \dim Ge$ $\forall s \in S$. Denote this number by d.

We have a morphism $S \longrightarrow Gr(d, V)$, $s \mapsto T_s Gs$. The locus $\{U \in Gr(d,V) \mid U \oplus S_0 = V\} \subset Gr(d,V)$ is open. We have $T_s S = S_0$ $\forall s$. From here we conclude that $\{s \in S \mid T_s S \oplus T_s Gs = V\}$ is Zariski open in S finishing the proof $\square$

3) Completion of the proof

As advertised in Sec 2.4 of Lec 10, we need two claims. We write π_S for $\pi|_S$ & 0 for $\pi(0)$.

Lemma 1: $\pi_S: S \to V/\!/G$ is finite.

Proof:

The actions of $\mathbb{C}^\times$ on $\mathbb{C}[S]$, $\mathbb{C}[V/\!/G]=\mathbb{C}[V]^G$ equip these algebras w. gradings, say $\mathbb{C}[S]_i := \{f\in \mathbb{C}[S] \mid t.f = t^{-i}f\}$. Since the actions are contracting, these gradings are positive (e.g. $\mathbb{C}[S]=\bigoplus_{i\geqslant 0}\mathbb{C}[S]_i$ & $\mathbb{C}[S]_0=\mathbb{C}$). Let $\mathfrak{m}=\bigoplus_{i>0}\mathbb{C}[V/\!/G]_i$ be the maximal ideal of 0 in $\mathbb{C}[V/\!/G]$. Now recall (Corollary 1 in Sec 2) that $\pi_S^{-1}(0)=\{e\}$. In particular, $\mathbb{C}[S]/\mathbb{C}[S]\mathfrak{m}$ is finite dimensional. A graded version of the Nakayama lemma implies that $\mathbb{C}[S]$ is a finitely generated module over $\mathbb{C}[V/\!/G]$ (details are left as an exercise) finishing the proof $\square$

Lemma 2: π_S is etale outside of codim 2, i.e.

$\operatorname{codim}\{s\in S \mid d_s\pi_S \text{ is not iso}\}\geqslant 2$

Proof:

Consider the morphism $\alpha: G\times S\to V$, $(g,s)\mapsto gs$. Thx to Corollary 2 in Sec 2, α is smooth (exercise) in particular all fibers have the same dimension. Combining the smoothness of α w. Claim in Sec 1 we see that the locus

$$\{(g,s) \mid d_{(g,s)}(\pi\circ\alpha) \text{ is not surjective}\} \subset G\times S$$

has codimension $\geqslant 2$. But $[\pi\circ\alpha](g,s) = \pi_S(s)$. This implies the claim of Lemma. $\square$

Now we are ready to finish the proof. The morphism $\pi_S: S \to V/\!/G$ between isomorphic affine spaces is finite & etale outside of codim 2. Since an affine space is strongly simply connected any such morphism is an isomorphism.