Invariant theory, Lec 11, 2/13/25.

1) Hilbert-Mumford theorem

Refs: [PV], Secs 5.3, 5.4.

1) Hilbert-Mumford theorem

1.0) Statement

We need the following construction. Let X be an affine variety over $\mathbb{C}$ equipped with an action of $\mathbb{C}^\times$. Pick $x\in X$. Then we have a morphism $\alpha: \mathbb{C}^\times \longrightarrow X$, $t \mapsto t\cdot x$. This morphism admits at most one extension $\bar{\alpha}: \mathbb{C}\to X$. If $\bar{\alpha}$ exists we say that $\lim_{t\to 0} t.x$ exists and equals $\bar{\alpha}(0)$. Of course, $\bar{\alpha}(0)$ is equal to the limit in the sense of the usual topology.

Our goal for this lecture is to prove the following result

Thm (Mumford): Let G be a reductive group acting on an affine variety X. Let $x\in X$ and $y \in X$ be such that Gy is the unique closed G-orbit in $\overline{Gx}$. Then $\exists$ an algebraic group homomorphism $\gamma: \mathbb{C}^\times \to G$ (such homomorphisms are known as one-parameter subgroups) s.t. $\lim_{t\to 0}\gamma(t)x$ exists and lies in Gy.

In the next lecture we'll consider some applications to checking the closedness of orbits & computing fibers of $X \to X/\!/G$.

We'll need the following fact which follows easily from the claim that irreducible varieties are connected in the usual topology.

Fact: 1) Let X be an irreducible variety $/\mathbb{C}$ & X° be an open subvariety. Then $X^\circ \subset X$ is dense in the usual topology

2) As a corollary, if X is any variety & $Y \subset X$ a (Zariski) locally closed subvariety. Then the closures of Y in Zariski & usual topologies coincide.

Hence the condition $Gy \subset \overline{Gx}$ means $\exists\, g_i \in G, i \geqslant 0$, such that the limit $\lim\limits_{k\to\infty} g_k x$ (in the usual topology) exists and lies in Gy. The theorem says that one can choose the g_i's in a subgroup of G isomorphic to $\mathbb{C}^\times$ making taking the limit much more controllable.

The theorem is proved in 2 steps. First, we analyze the case when G is a torus and then reduce the general case to this one.

1.1) Case of torus

Let $G = T$ be a torus (so that $T \xrightarrow{\sim} (\mathbb{C}^\times)^n$) & V be its finite dimensional rational representation, completely reducible b/c T is reductive.

1.1.1) Support & polytopes. Any irreducible representation of T is 1-dimensional, hence given by a **character** i.e. an algebraic group homomorphism $T \longrightarrow GL_1(\mathbb{C}) = \mathbb{C}^\times$. Such characters form a group called the **character lattice** of T & denoted by $\mathfrak{X}^*(T)$. Indeed under an identification of T with $(\mathbb{C}^\times)^n$ the characters are given by $(t_1, \dots, t_n) \mapsto \prod_{i=1}^{n} t_i^{d_i}$ for unique $d_1, \dots d_n \in \mathbb{Z}$ giving a group isomorphism $\mathfrak{X}(T) \xrightarrow{\sim} \mathbb{Z}^n$.

Now let $v \in V$. We can uniquely write v

$$v = \sum_{\chi \in \mathfrak{X}^*(T)} v_\chi \quad \text{w.} \quad t.v_\chi = \chi(t) v$$

(note that sum has only finitely many nonzero summands).

Definition: 1) By **support** of v we mean the set

$$\operatorname{Supp}(v) := \{\chi \in \mathfrak{X}^*(T) \mid v_\chi \neq 0\}$$

2) Denote by $\operatorname{Conv}(v)$ the convex hull of $\operatorname{Supp}(v)$ i.e

$$\operatorname{Conv}(v) = \Big\{ \sum_{\chi \in \operatorname{Supp}(v)} a_\chi \chi \;\Big|\; a_\chi \geq 0 \;\&\; \sum a_\chi = 1 \Big\} \subset t^*_{\mathbb{R}} \;(= \mathbb{R} \otimes_{\mathbb{Z}} \mathfrak{X}^*(T))$$

and by $\operatorname{Int}(v)$ the relative interior of $\operatorname{Conv}(v)$.

$$\operatorname{Int}(v) = \Big\{ \sum_{\chi \in \operatorname{Supp}(v)} a_\chi \chi \;\Big|\; a_\chi > 0 \;\&\; \sum a_\chi = 1 \Big\}$$

It turns out that one can extract a lot of useful info from these invariants. E.g:

Exercise: $\dim Tv = \dim \mathrm{Span}_{\mathbb{R}}(\mathrm{Supp}(v))$.
Hint: $\mathrm{Stab}_T(v) = \bigcap_{\chi \in \mathrm{Supp}(v)} \ker \chi$.

1.1.2) Invariants

Choose a basis $v_1, \ldots v_n$ of eigenvectors for T w. eigenvalues $\chi_1, \ldots \chi_n \in \mathfrak{X}^*(T)$. Let $x_1, \ldots x_n \in V^*$ be the dual basis. Then T acts on the monomial $x_1^{d_1} \ldots x_n^{d_n}$ by character $-\sum_{i=1}^n d_i \chi_i$. In particular we have

Lemma: $\mathbb{C}[V]^T \subset \mathbb{C}[V]$ is the span of all monomials $x_1^{d_1} \ldots x_n^{d_n}$ w. $\sum_{i=1}^n d_i \chi_i = 0$.

Bonus remark: A nice feature of the variety $V/\!\!/T$ is that it is toric. In general, a toric variety is a normal variety X equipped with an action of a torus H that has an open orbit in X. These varieties are important for Algebraic geometry b/c they can be understood completely combinatorially. It turns out that $V/\!\!/T$ is an example. Namely let $\tilde{T}$ be the subgroup of diagonal matrices in $GL(V)$ w.r.t. the basis $v_1, \ldots v_n$. Then $T \subset \tilde{T}$ & $\tilde{T}/T$

acts on $V/\!/T$ turning the latter into a toric variety. Basically, any affine toric variety can be obtained in this way.

1.1.3) Closed orbits

Proposition: Let $v \in V$. If $0 \in Int(v)$, then Tv is closed.

In fact, the converse is true as well but we don't need this.

Proof:

Let $Supp(v) = \{\chi_1, \dots \chi_n\}$ so that $v = \sum_{i=1}^{n} v_i$ w. $t.v_i = \chi_i(t) v_i$ & $v_i \neq 0$. Note that $Tv \subset Span(v_i \mid i = 1, \dots n)$, which is a T-stable subspace. Clearly, Tv is closed in V iff it's closed in this subspace so we can assume that $V = Span(v_i)$. Since the v_i's are linearly independent (eigenvectors w. pairwise distinct e-values) they form a basis in V. Let $x_1, \dots x_n \in V^*$ be the dual basis.

The condition $0 \in Int(V)$ is equivalent to: $\exists\, \alpha_1, \dots \alpha_n \in \mathbb{Z}_{>0} \mid \sum_{i=1}^{n} \alpha_i \chi_i = 0 \Rightarrow$ [Lemma in Sec 1.1.2] $f = x_1^{\alpha_1} \dots x_n^{\alpha_n} \in \mathbb{C}[V]^T$.

Note that $x_i(v) = 1 \Rightarrow f(v) = 1$.

Now suppose $Tu \subset \overline{Tv} \Rightarrow f(u) = 1 \Rightarrow [u = \sum a_i v_i]\ \prod_{i=1}^{n} a_i^{\alpha_i} = 1$ $\Rightarrow a_i \neq 0\ \forall\, i \Leftrightarrow Supp(u) = Supp(v)$

Now we use Exercise in Sec 1.1.1 to deduce that $\dim Tu = \dim Tv \Rightarrow Tu = Tv$ proving Tv is closed. $\square$

1.1.4) Hilbert-Mumford for tori

Proof of Theorem for $G=T$:

Step 1: Here we reduce to the case when X is a rational representation. First assume that G is general algebraic group acting on an affine variety X. By Sec 1.2 in Lec 3, there is a finite dimensional rational subrepresentation $V'\subset \mathbb{C}[X]$ w. $S(V')\twoheadrightarrow \mathbb{C}[X] \Leftrightarrow X\hookrightarrow V$ where $V'=V^*$.

Apply this for $G=T$. For $x\in X$, the closure of Gx in X is the same as the closure in V. So we can replace X w. V.

Our strategy in subsequent steps is as follows: from $v\in V$ we construct a vector $u\in V$ w. closed Tu & $\gamma: \mathbb{C}^\times \to T$ w. $\lim_{t\to 0}\gamma(t)v=u$ (implying the claim of Thm for $G=T$ b/c $\overline{Tv}$ contains a unique closed orbit).

Step 2: Here we construct $u\in V$. Note that any convex polytope has at most one face whose relative interior contains 0. We need to treat two cases separately.

Case 1: $\exists$ face, F, of $Conv(v)$ whose interior contains 0. Observe that F equals the convex hull of $F_v := Supp(v)\cap F$. Set

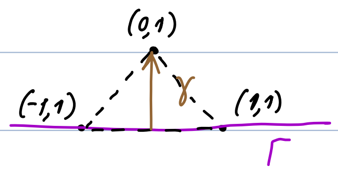
$$u=\sum_{\chi\in F_v} v_\chi$$

Then $\mathrm{Conv}(u)=F$, so $0\in \mathrm{Int}(u)$. By Proposition in Sec. 1.1.3, T_u is closed.

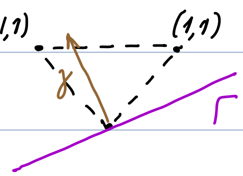
Case 2: $\mathrm{Conv}(v)$ has no such face. Set $u:=0 \Rightarrow T_u$ is closed.

Step 3: We construct $\gamma: \mathbb{C}^\times \to T$. Such homomorphisms form a lattice denoted by $\mathfrak{X}_*(T)$. It has a natural pairing $\mathfrak{X}_*(T)\times \mathfrak{X}^*(T) \longrightarrow \mathbb{Z}$ given by $\chi(\gamma(t)) = t^{\langle \gamma,\chi\rangle}$ $\forall\, t\in \mathbb{C}^\times$. This pairing is perfect (exercise). Note that $\mathfrak{X}_*(T)$ is a $\mathbb{Z}$-lattice in $\mathfrak{t}$. Here's how γ is constructed. If $0\in \mathrm{Int}(v)$ take γ to be trivial. Otherwise we do the following:

Case 1: We can find a rational (linear) hyperplane $\Gamma\subset \mathfrak{t}_{\mathbb{R}}$ s.t. $\Gamma\cap \mathrm{Conv}(v) = F$ & $\mathrm{Conv}(v)$ lies to one side of Γ. E.g.

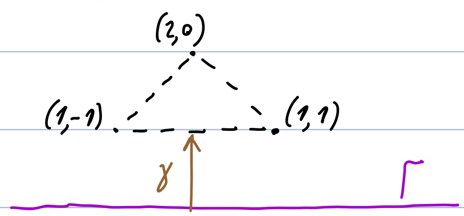


or



Take $\gamma\in \mathfrak{X}_*(T)$ s.t. $\Gamma = \ker\gamma$ & $\gamma\geq 0$ on $\mathrm{Conv}(v)$

Case 2: Similarly, we can find γ s.t. γ is positive on $\mathrm{Conv}(v)$, e.g.



Step 4: We claim that $\lim_{t\to 0} \gamma(t)v = u$. Note that if $\chi \in Supp(v)$, then $\langle \gamma,\chi\rangle \geqslant 0$ w. equality iff $\chi\in F_v$ (declared to be empty in Case 2). Then

$$\gamma(t)v = \sum_{\chi\in Supp(v)} t^{\langle\chi,\gamma\rangle} v_\chi$$

Now we are done by the following important & easy exercise.

Exercise: Suppose $\mathbb{C}^\times$ acts on a vector space V & for $i\in\mathbb{Z}$ we set $V_n = \{v\in V \mid t.v = t^n v\}$. Write $v\in V$ as $\sum_n v_n$ w. $v_i \in V_i$. Then $\lim_{t\to 0} t.v$ exists iff $v_i = 0$ for all $i<0$ and in this case it equals v_0.

$\square$

1.2) Case of general G.

Here G is connected reductive group. We will give a proof that only works over $\mathbb{C}$.

1.2.1) Cartan decomposition.

Fact 1 (see [OV], Sec 5.1) $\exists$ antiholomorphic involution $\sigma: G\to G$ s.t. $K := G^\sigma$ is compact (and is maximal compact subgroup w.r.t. inclusion - in fact, it's Zariski dense). Moreover, we can find a σ-stable maximal torus $T\subset G$.

Example: Let $G = GL(n)$. Then $\sigma(g) = (g^*)^{-1}$, where $g^* = \bar{g}^T$. We have $K = U(n)$. For T we can take the subgroup of diagonal matrices.

Theorem (Cartan decomposition)

$G = KTK \ (= \{k_1 t k_2 \mid k_1, k_2 \in K, t \in T\})$.

Proof for $G = GL_n$:

From Linear algebra we know "polar decomposition": $H_+ \times U_n \xrightarrow{\sim} GL_n$, $(h,u) \mapsto hu$, where H_+ is the subset of positive definite Hermitian matrices. By Spectral theorem, $\forall h \in H_+$ $\exists k \in U(n)$, $t \in T \cap H_+$ w. $h = ktk^{-1}$. So any $g \in GL(n)$ can be written as $hu = kt(k^{-1}u)$ yielding the claim of Thm $\square$

We will provide more details on Cartan decomposition in Bonus Section 1.2.3.

1.2.2) Proof of Hilbert-Mumford theorem

Thanks to Fact 2 from Sec 1.0, we can deal w. the usual topology (instead of Zariski topology)

The action map $K \times X \to X$ is proper (preimage of compact is compact) hence closed. Hence $\overline{Gx} = K\,\overline{TKx} \Rightarrow \overline{TKx} \cap Gy \neq \emptyset$ $\Rightarrow \exists$ sequences $t_i \in T$, $k_i \in K$ $(i \geq 0)$ s.t. $\lim\limits_{i \to \infty} t_i k_i x = hy$ $(h \in G)$.

Since K is compact can replace (k_i) w. a subsequence & assume $\lim_{i\to\infty} k_i = k$ for some $k\in K$.

Now consider the quotient morphism $\pi_T: X\to X/\!/T$. We have

$$\pi_T(hy)=\lim_{i\to\infty}\pi(t_ik_ix)=[\pi_T(t_ik_ix)=\pi_T(k_ix)]=\lim_{i\to\infty}\pi_T(k_ix)=\pi_T(kx).$$

Let Ty' be the unique closed T-orbit in $\pi_T^{-1}(\pi_T(hy))$. Then $Ty'\subset\overline{Ty}\subset[Gy$ is closed$]\subset Gy$. Also $Ty'\subset\overline{Tkx}$. By Sec 1.1.4, $\exists\,\gamma:\mathbb{C}^\times\to T$ s.t. $\lim_{t\to 0}\gamma(t)kx\in Ty'\subset Gy$. This implies the theorem. $\square$

Rem: For an algebraic proof see [MF], Sec 2.1.

1.2.3) Bonus: Cartan decomposition for general reductive groups.

A reference here is [OV], Sec 5.2. A key fact is that for any connected reductive algebraic subgroup $G\subset GL(V)$ there's a Hermitian scalar product on V s.t. G is closed under $g\mapsto g^*$. Then one can set σ from Sec 1.2.1 to be $g\mapsto (g^*)^{-1}$. Then one shows that exp defines a diffeomorphism

(*) $\sqrt{-1}\,\mathfrak{k}\xrightarrow{\sim} H_+(V)\cap G$ positive definite Hermitian operators in $\mathrm{End}(V)$.

From here we see that every element of $H_+(V)\cap G$ has a unique square root in $H_+(V)\cap G$. Then we get polar decomposition: for $g\in G$ we have $gg^*\in H_+(V)\cap G$ & we can write

$$g = \sqrt{gg^*}\left(\sqrt{gg^*}^{-1} \cdot g\right), \quad \sqrt{gg^*} \in H_+(V)$$

Moreover, every element of $\mathfrak{k}$ is K-conjugate to an element of $\mathfrak{k} \cap \mathfrak{t}$, where $\mathfrak{t}$ is a σ-stable Cartan. This gives a generalization of the spectral theorem for $H_+(V) \cap G$ thx to $(*)$.

With these ingredients the proof of Cartan decomposition for general G repeats that for $G = GL_n$.