Invariant theory, Lec 12, 2/19/25.

1) Applications of Hilbert-Mumford theorem.
2) Optimal destabilizing subgroups.

Ref: [PV], Secs 5.4 & 5.5

1) Applications of Hilbert-Mumford theorem.

In Lecture 11 we have proved:

Thm (Mumford): Let G be a reductive group acting on an affine variety X. Let $x\in X$ and $y\in X$ be such that Gy is the unique closed G-orbit in $\overline{Gx}$. Then $\exists$ a one-parameter subgroup $\gamma: \mathbb{C}^\times \longrightarrow G$ s.t. $\lim_{t\to 0}\gamma(t)x$ exists and lies in Gy.

We are going to apply this theorem to determine:

(i) $\pi^{-1}(\pi(0))$ for certain rational representations $G \curvearrowright V$.

(ii) the closed orbits in V, and, more generally, the closed orbit in the closure of a given one.

1.1) One-parameter subgroups.

We start by studying the structure of 1-param. subgroups to G up to G-conjugacy. Let T be a maximal torus in G, W be the Weyl group $N_G(T)/T$. We fix a Borel subgroup $B\subset G$, which gives us the positive Weyl chamber $C\subset \mathfrak{h}_{\mathbb{R}}$, where $\mathfrak{h}=\mathrm{Lie}(T)$. Note that the lattice of 1-parameter subgroups $\mathfrak{X}_*(T)$ is a $\mathbb{Z}$-lattice in $\mathfrak{h}_{\mathbb{R}}$.

$\forall\ \gamma:\mathbb{C}^\times\to G \Rightarrow \gamma(\mathbb{C}^\times)$ is a torus $\Rightarrow \gamma(\mathbb{C}^\times)\subset$ a max. torus $\Rightarrow \gamma$ is G-conjugate to an element of $\mathfrak{X}_*(T)$. It's defined uniquely up to W-conjugation $\Rightarrow \exists!$ one-parameter subgroup in $\mathfrak{X}_*(T)\cap C$ that is G-conjugate to γ. So we just consider $\gamma\in\mathfrak{X}_*(T)\cap C$.

Also note that $\lim\limits_{t\to 0}\gamma(t)x=\lim\limits_{t\to 0}\gamma^k(t)x\ \ \forall\, k\in\mathbb{Z}_{>0}$. So it's enough to consider elements of $\mathfrak{X}_*(T)\cap C$ up to a positive integer multiple.

Example: Let $G=SL_2$ so $T=\{\mathrm{diag}(t,t^{-1})\mid t\in\mathbb{C}^\times\}$. We have $\mathfrak{X}_*(T)=\mathbb{Z}$ & $\mathfrak{X}_*(T)\cap C=\mathbb{Z}_{\geq 0}$. Since the trivial 1-param. subgroup doesn't give anything interesting, it's enough to consider $\gamma: t\longmapsto \mathrm{diag}(t,t^{-1})$.

1.2) Action of SL_2 on $S^n(\mathbb{C}^2)$

This is an example considered by Hilbert. Here we can answer questions (i) & (ii) completely.

Let γ be as in Example in Sec 1.1. We decompose $V=S^n(\mathbb{C}^2)$ as $\bigoplus\limits_{i=0}^{n}V_{2i-n}$, where $V_{2i-n}=\{v\in V\mid \gamma(t)v=t^{2i-n}v\}=\mathbb{C}\cdot x^iy^{n-i}$ for the standard basis x, y of $\mathbb{C}^2$. We write $v = \sum v_j$ w. $v_j \in V_j$. As was remarked already in Lec 1, every $v \in V$ viewed as a homogeneous deg n polynomial in x, y decomposes into the product of linear factors essentially uniquely.

$\lim_{t\to 0} \gamma(t)v$ exists iff $v_j = 0\ \forall j<0$ ($\Leftrightarrow v$ is divisible by $x^{\lceil n/2 \rceil}$) & equals v_0 (i.e. a multiple of $x^{n/2}y^{n/2}$ if n is even and 0 else).

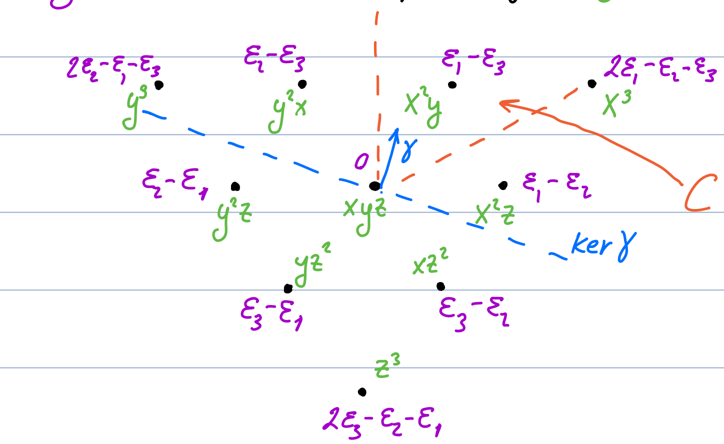
Let's describe $\pi^{-1}(\pi(0))$. Hilbert-Mumford theorem & the above analysis shows that $\pi^{-1}(\pi(0)) = G.V_{>0}$, where $V_0 = \bigoplus_{i=\lceil n/2 \rceil}^{n} V_{2i-n}$. The elements of $G.V_{>0}$ can be characterized as those that have a linear factor with multiplicity $> n/2$.

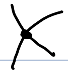
Let's describe the other fibers & closed orbits in them. The above analysis shows that if all linear factors of v have multiplicities $< \frac{n}{2}$, then $\lim_{t\to 0} \gamma(t)v$ doesn't exist $\forall$ non-trivial γ. The Hilbert-Mumford theorem shows that Gv is closed.

The only case remaining is when $\exists$ linear factor of v w. multiplicity exactly $n/2$. The above analysis shows that if for a 1-parameter subgroup γ the limit $\lim_{t\to 0} \gamma(t)v$ exists, then it has two linear factors both w. multiplicity $\frac{n}{2}$. From here it follows that any orbit of such an element (containing an element of the form $cx^{n/2}y^{n/2}$ for $c \in \mathbb{C}^\times$) is closed (exercise; hint: apply Hilbert-Mumford to $x = v$). And the orbit of any element with exactly one factor w. multiplicity $\frac{n}{2}$ is not closed – it contains an orbit $G.(c\,x^{n/2}y^{n/2})$ for suitable $c\in\mathbb{C}$ in the closure.

1.3) Action of SL_3 on $S^3(\mathbb{C}^3)$

This is a more difficult case as there are many more 1-param. subgroups. We will only describe $\pi^{-1}(\pi(0))$. Let $V=\bigoplus_\lambda V_\lambda$ denote the weight decomposition. All weight spaces are 1-dimensional, the weights and the corresponding weight vectors are depicted here:



Hilbert-Mumford theorem says: $v\in\pi^{-1}(\pi(0))$ iff $\exists\ \gamma\in\mathfrak{X}_*(T)\cap C$ s.t. $v\in G.\left(\bigoplus_{\lambda\,|\,\langle\lambda,\gamma\rangle>0} V_\lambda\right)$. It is easy to see that $\exists!$ max'l (w.r.t. $\subseteq$) subset of the form $\{\lambda\,|\,\langle\gamma,\lambda\rangle>0\}$, it is $\{2\varepsilon_2-\varepsilon_1-\varepsilon_3,\ \varepsilon_2-\varepsilon_3,\ \varepsilon_1-\varepsilon_3,\ 2\varepsilon_1-\varepsilon_2-\varepsilon_3,\ \varepsilon_1-\varepsilon_2\}$. So $\pi^{-1}(\pi(0))$ consists of the conjugates of polynomials of the form $ay^3+by^2x+cx^2y+dx^3+ex^2z$. Such non-zero polynomials are characterized by the following property of the corresponding curve in $\mathbb{P}^2$: the point $[0:0:1]$ is neither smooth, nor ordinary double points:  . So the nonzero points in $\pi^{-1}(\pi(0))$ correspond precisely to the deg 3 curves in $\mathbb{P}^2$ which have singularities different from ordinary double points.

1.4) Action of $GL(V)$ on $End(V)^{\oplus \ell}$.

We proceed to an example of a different nature: $GL(V)$ acting on $End(V)^{\oplus \ell}$ by conjugations. We are going to view elements of $End(V)^{\oplus \ell}$ as representations of the <u>set</u> $I = \{1, \dots, \ell\}$. The standard terminology applies: we can talk about irreducible & completely reducible representations, filtrations on a representation, and, in particular, a Jordan-Hölder (JH) filtration (one that cannot be refined). Note that the $GL(V)$-orbits are exactly isomorphism classes of representations.

Now let $\{0\} = V_0 \subsetneq V_1 \subsetneq \dots \subsetneq V_k = V$ be a filtration of a representation of I, denote it by F. We can form the associated representation, $gr^F V := \bigoplus_{i=1}^{k} V_i / V_{i-1}$. We can identify $gr^F V$ with V as vector spaces and view the associated graded as an element of $End(V)^{\oplus \ell}$, this gives a well-defined $GL(V)$-orbit (but not an individual element).

Proposition: Let $\underline{A} := (A_1, \dots A_\ell) \in End(V)^{\oplus \ell}$ be a representation of I.

(1) $GL(V).\underline{A}$ is closed $\Leftrightarrow$ $\underline{A}$ is completely reducible

(2) For general $\underline{A}$, the unique closed orbit in $\overline{GL(V).\underline{A}}$ corresponds to the associated graded of $\underline{A}$ w.r.t. any JH filtration.

Proof: Choose a basis $e_1, e_2, \dots, e_n \in V \rightsquigarrow End(V) \xrightarrow{\sim} Mat_n(\mathbb{C})$.

Step 1: Let's examine when the limit $\lim_{t\to 0} \gamma(t).\underline{A}$ exists & what it equals to. We assume that $\gamma \in \mathfrak{X}_*(T) \cap C$, which translates to $\gamma(t) = diag(\underbrace{t^{d_1}, \dots t^{d_1}}_{n_1}, \underbrace{t^{d_2}, \dots t^{d_2}}_{n_2}, \dots, t^{d_m}, \dots t^{d_m})$, $d_1 > d_2 > \dots > d_m$. Of course,

$$\lim_{t\to 0} \gamma(t).\underline{A} = (\lim_{t\to 0} \gamma(t) A_1 \gamma(t)^{-1}, \dots, \lim_{t\to 0} \gamma(t) A_\ell \gamma(t)^{-1})$$

We write A_i as (b_{jk}) w. $b_{jk} \in Mat_{d_j \times d_k}(\mathbb{C})$. Then $\gamma(t) A_i \gamma(t)^{-1} = (t^{d_j - d_k} b_{jk})$

It follows that $\lim_{t\to 0} \gamma(t) A_i \gamma(t)^{-1}$ exists $\iff b_{jk} = 0$ for $j > k$, & if so it equals $diag(b_{jj})$. In the language of representations, this just says that $V_1 = Span_{\mathbb{C}}(e_1, \dots e_{n_1}) \subset V_2 = Span_{\mathbb{C}}(e_1, \dots, e_{n_1+n_2}) \subset \dots$ gives a filtration for $\underline{A}$ and the limit $\lim_{t\to 0} \gamma(t).\underline{A}$ is the associated graded representation.

Step 2: Note that a representation is completely reducible iff it's isomorphic to its associated graded with respect to any filtration. This immediately implies that "closed orbit $\Rightarrow$ completely reducible" using Step 1. "Completely reducible $\Rightarrow$ closed orbit" follows from Step 1 thx to Hilbert-Mumford: complete reducibility implies: $\forall\ \gamma: \mathbb{C}^\times \to G$ s.t. $\lim_{t\to 0} \gamma(t).\underline{A}$ exists, this limit is conjugate to $\underline{A}$. To prove (2) is an exercise. □

Rem: Let $\mathcal{A}$ be a finitely generated associative algebra with generators $x_1, \dots x_\ell$. Then the homomorphism $\mathcal{A} \hookrightarrow End(V)$ form a closed subscheme in $End(V)^{\oplus \ell}$ to be denoted by $Rep(\mathcal{A}, V)$ (we look at the images of x_i's). This subscheme is $GL(V)$-stable & so we have a closed inclusion $Rep(\mathcal{A}, V)/\!/GL(V) \hookrightarrow End(V)^{\oplus \ell}/\!/GL(V)$. The $\mathbb{C}$-points of $Rep(\mathcal{A}, V)/\!/GL(V)$ parameterize completely reducible representations of $\mathcal{A}$ (up to isomorphism).

2) Optimal destabilizing subgroups.

Let G be a connected reductive group acting on its finite dimensional representation V. Let $\pi: V \to V/\!/G$ denote the quotient morphism.

The goal of this part is to better understand the structure of $\pi^{-1}(\pi(0))$. Namely we will introduce a stratification on $\pi^{-1}(\pi(0))$ called the Kirwan-Ness stratification. The strata are labelled by G-conjugacy classes of (basically) 1-param. subgroups $\gamma: \mathbb{C}^\times \to G$: for a point labelled by γ, a G-conjugate of γ is the 1-param. subgroup sending the element to 0 in the fastest possible way (a.k.a. optimal destabilizing subgroup).

Our first task is to try to make this notion formal. We fix a nondegenerate form $(\cdot,\cdot)$ on $\mathfrak{g}$ s.t. its restriction to $\mathfrak{h}_{\mathbb{Q}}$ takes rational values. For example, we can pick a faithful representation U of $\mathfrak{g}$ and set $(x,y):=\mathrm{tr}_U(xy)$. For $\gamma:\mathbb{C}^\times\to G$, let $h_\gamma:=d_1\gamma(1)$, so that h_γ is conjugate to an element of $\mathfrak{h}_{\mathbb{Q}}$.

Now take $v\in V\setminus\{0\}$ w. $\pi(v)=0$. By Hilbert-Mumford $\exists$ 1-param. subgroup $\gamma:\mathbb{C}^\times\to G$ w. $\lim_{t\to 0}\gamma(t)v=0$. Equivalently, we can set $V_n(\gamma)=\{v\in V \mid \gamma(t)v=t^n v \Leftrightarrow h_\gamma v=nv\}$, then $\lim_{t\to 0}\gamma(t)v=0 \Leftrightarrow v\in\bigoplus_{n>0}V_n(\gamma)$. For $h\in G.\mathfrak{h}_{\mathbb{Q}}$ & $a\in\mathbb{Q}$ define $V_a(h)$ similarly.

Definition/Lemma: $\forall v\in\pi^{-1}(\pi(0))\setminus\{0\}$ $\exists$ $h\in G.\mathfrak{h}_{\mathbb{Q}}$ s.t.

$$(*)\qquad v\in\bigoplus_{a\geq 2}V_a(h)$$

& (h,h) is minimal possible s.t. $(*)$ holds.

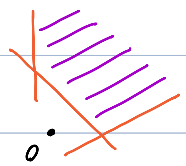
The element h is called the characteristic of v.

Moreover, the number of characteristics of nonzero elements of $\pi^{-1}(\pi(0))$ is finite (up to G-conjugacy).

A precise connection to "optimal destabilizing subgroups" is as follows. For γ & $v\neq 0$ s.t. $\lim_{t\to 0}\gamma(t)v=0$, let $e(\gamma,v)$ denote the minimal number $e>0$ w. $v\in\bigoplus_{n\geq e}V_\gamma(n)$. Then we are looking at minimizing $(\gamma,\gamma)^{1/2}/e(\gamma,v)$ (which doesn't change when we replace γ w. a positive multiple). This quantity is minimized when h_γ is proportional to a characteristic.

Proof: This is an exercise in Convex geometry.

It suffices to look only at $h\in\mathfrak{h}_{\mathbb{Q}}$ (everything is up to G-conjugacy). Restrict $G \curvearrowright V$ to T. Consider the set $\mathcal{P}$ of all possible $\mathrm{Conv}(v)\subset\mathfrak{h}^*_{\mathbb{R}}$ for $v\in V$; $\lim_{t\to 0}\gamma(t)v=0$ for $\gamma\in\mathfrak{X}_*(T) \Leftrightarrow 0\notin \mathrm{Supp}(v)$. The set of all $h\in\mathfrak{h}_{\mathbb{R}}$ s.t. $v\in\bigoplus_{a\geq 1}V_h(a)$ is the set of all linear functions $\mathfrak{h}^*_{\mathbb{R}}\to\mathbb{R}$ that take values ≥ 1 on the vertices of $\mathrm{Conv}(v)$. It looks smth. like this: an (unbounded) convex region defined by rational inequalities not containing 0.



Exercise: any such contains a unique point with minimal length & this point is rational.

Now we can find a characteristic of v as follows. Consider all $P_1,\dots P_m\in\mathcal{P}$ s.t. $P_i=\mathrm{Supp}(gv)$ for $g\in G$ & $0\notin P_i$. There are only finitely many as P_i is determined a subset in the set of weights of V & the latter is finite.

From P_i produce a unique element $h_i\in\mathfrak{h}_{\mathbb{Q}}$ as in Exercise.

For a characteristic of v we can take h_i, $i=1,\dots m$, of minimal length.

And since $\mathcal{P}$ is a finite set, the number of possible characteristics is also finite (up to G-conjugacy) $\square$