Lecture 13, 2/24/25

1) Optimal destabilizing subgroups.
2) Hesselink/Kirwan-Ness stratification.

Ref: [PV], Secs 5.5, 5.6.

1) Optimal destabilizing subgroups.

1.0) Reminder/notation.

Let G be a connected reductive group$/\mathbb{C}$, V be its fin. dim. rational representation, $\pi: V \longrightarrow V/\!\!/G$ be the quotient morphism.

Choose a maximal torus $T\subset G$ & let $\mathfrak{h}$ be its Lie algebra. Set $\mathfrak{h}_{\mathbb{Q}} = \mathbb{Q}\otimes_{\mathbb{Z}} \mathfrak{X}_*(T)$. For $h\in G.\mathfrak{h}_{\mathbb{Q}}\setminus\{0\}$ consider the $\mathbb{Q}$-gradings:

$$V=\bigoplus_{a\in\mathbb{Q}} V_a(h),\ \mathfrak{g}=\bigoplus_{a\in\mathbb{Q}}\mathfrak{g}_a(h), \text{ where say } V_a(h)=\ker(h-a\,\mathrm{id}_V).$$

For $b\in\mathbb{Q}$ we set $V_{\geqslant b}(h)=\bigoplus_{a\geqslant b} V_a(h)$ & let $\mathfrak{g}_{\geqslant b}(h)$ have the similar meaning. Note that:

- $\mathfrak{g}_0(h)\subset\mathfrak{g}_{\geqslant 0}(h)\subset\mathfrak{g}$ are subalgebras; $\exists$ connected algebraic subgroups $G_0(h)\subset G_{\geqslant 0}(h)\subset G$ with these Lie algebras. The subgroup $G_0(h)$ (a Levi subgroup) is reductive, see Case 2 in Sec 1.1 of Lec 7, while $G_{\geqslant 0}(h)$ is a "parabolic" subgroup, which by definition means that it contains a Borel subgroup. The projection $\mathfrak{g}_{\geqslant 0}(h)\twoheadrightarrow\mathfrak{g}_0(h)$ integrates to $G_{\geqslant 0}(h)\twoheadrightarrow G_0(h)$.
- $V_a(h)$ is $G_0(h)$-stable & $V_{\geqslant a}(h)\subset V_{\geqslant a}(h)$ are $G_{\geqslant 0}(h)$-stable $\forall a$.

Moreover, the action on $G_{\geq 0}(h)$ on $V_{\geq a}(h)/V_{>a}(h) \xrightarrow{\sim} V_a(h)$ factors through $G_0(h)$.

Fix a non-degenerate G-invariant symmetric form $(\cdot,\cdot)$ on $\mathfrak{g}$ w. $(x,x)\in\mathbb{Q}_{\geq 0}$ $\forall x\in\mathfrak{h}_{\mathbb{Q}}$. We said in Sec. 2 of Lec 12 that $h\in G.\mathfrak{h}_{\mathbb{Q}}$ is a characteristic of $v\in\pi^{-1}(\pi(0))\setminus\{0\}$ if $v\in V_{\geq 2}(h)$ & (h,h) is min. possible with this property, in particular, this notion depends on the choice of $(\cdot,\cdot)$. We have seen that a characteristic exists.

Here's a basic result linking the characteristic to the subgroups introduced above.

Exercise: if h is a characteristic of v & $g\in G_{\geq 0}(h)$, then

i) h is a characteristic of gv (hint: this follows from the invariance of $(\cdot,\cdot)$ & the claim that $G_{\geq 0}(h)$ preserves $V_{\geq 2}(h)$).

ii) gh is a characteristic of v.

We will see below any characteristic of v is of the form $g.h$ for $g\in G_{\geq 0}(h)$ & h being a characteristic of v.

1.1) Main results.

Let $h\in G.\mathfrak{h}_{\mathbb{Q}}\setminus\{0\}$.

We need a certain normal subgroup of $G_0(h)$. Namely, h is in the center of $\mathfrak{g}_0(h)$. Set $\underline{\mathfrak{g}}_0(h) = \{x \in \mathfrak{g}_0(h) \mid (h,x)=0\}$. This is an ideal. Moreover, it's a Lie algebra of a normal subgroup of $G_0(h)$: $(h,\cdot)$ is a rational element of $(\mathfrak{g}_0(h)/[\mathfrak{g}_0(h),\mathfrak{g}_0(h)])^*$ hence some multiple of $(h,\cdot)$ comes from a homomorphism

$$\chi: G_0(h) \longrightarrow \mathbb{C}^\times.$$

Set $\underline{G}_0(h) := (\ker \chi)^\circ$. Note that $\underline{G}_0(h)$ acts on $V_2(h)$, let $\underline{\pi}_2 : V_2(h) \twoheadrightarrow V_2(h)/\!/\underline{G}_0(h)$ denote the quotient morphism. Finally, for $v \in V_{\geqslant 2}(h)$ we write v_2 for its component in $V_2(h)$.

Theorem (Kirwan; Ness)

1) Let $v \in V_{\geqslant 2}(h)$. TFAE
   a) h is a characteristic of v.
   b) $\underline{\pi}_2(v_2) \neq \underline{\pi}_2(0)$.
2) If h & h' are characteristics of v, then $h' \in G_{\geqslant 0}(h).h$.

Exercise: If h is a characteristic of v, then $\mathrm{Stab}_G(v) \subset G_{\geqslant 0}(h)$.

Example: Let $e \in \mathfrak{g}$ be a nonzero nilpotent element. Include it into an $\mathfrak{sl}_2$-triple (e,h,f), see Sec 2.3.1 in Lec 10. One can show that h is a characteristic of e: in fact, we will later see that

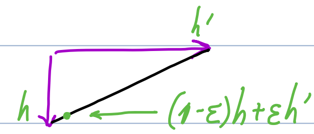
$\underline{G}_0(h).e$ is closed & since $e\in\mathfrak{g}_2(h)$, (b) of Thm yields the claim.

1.2) Proof of Theorem

We can assume $T\subset G_0(h) \Leftrightarrow \mathfrak{h}\subset\mathfrak{g}_0(h)$. Then $\underline{\mathfrak{h}} := \mathfrak{h}\cap\underline{\mathfrak{g}}_0(h)$ is a Cartan subalgebra in $\underline{\mathfrak{g}}_0(h)$. We write $|x|$ for $(x,x)^{1/2}$ $(x\in G.\mathfrak{h}_{\mathbb{R}})$

Step 1: We prove (a) $\Rightarrow$ (b). Suppose $\underline{\pi}_2(v)=\underline{\pi}_2(0)$. By Hilbert-Mumford, $\exists\ h'\in \underline{G}_0(h).\underline{\mathfrak{h}}_{\mathbb{Q}}$ s.t. $v_2\in V_{2,\geqslant 2}(h')$. Note that $[h,h']=0$. It follows that for $\varepsilon>0$ & $\varepsilon\ll 1$, $v\in V_{\geqslant 2}((1-\varepsilon)h+\varepsilon h')$.

Exercise: $|(1-\varepsilon)h+\varepsilon h'|<|h|$. Hint: $(h,h')=0$ &



This gives a contradiction w. (a).

Step 2: We want to give a "combinatorico-geometric" consequence of (b).

We start by introducing a bit more notation. The form $(\cdot,\cdot)$ induces an isomorphism $\mathfrak{h}\to\mathfrak{h}^*$ to be denoted by ι. We carry $(\cdot,\cdot)$ to $\mathfrak{h}^*$ using ι. Set $\chi=\frac{2\iota(h)}{(h,h)}$. This is the closest to 0 point on the hyperplane $\langle h,\cdot\rangle=2$ in $\mathfrak{h}^*_{\mathbb{R}}$.

To nonzero $u\in V$ we assign the polytope $\mathrm{Conv}(u)\subset\mathfrak{h}^*_{\mathbb{R}}$ from the action of $T \curvearrowright V$, see Sec. 1.1.1 of Lec 11.

Similarly, for nonzero $u \in V_2$ we have $\underline{\mathrm{Conv}}_2(u) \subset \underline{\mathfrak{h}}^*_{\mathbb{R}}$ (for the action of $\underline{T}$, the max. torus of $\underline{G}_0(h)$ w. $\mathrm{Lie}(\underline{T}) = \underline{\mathfrak{h}}$, on $V_2(h)$) & $\mathrm{Conv}_2(u) \subset \mathfrak{h}^*_{\mathbb{R}}$ for $T \curvearrowright V_2(h)$. Our goal is to prove the following:

Claim: Suppose (6) holds. Then

(1) $$\chi \in \mathrm{Conv}(gv) \ \forall g \in G_{\geqslant 0}(h).$$

Proof: By Hilbert-Mumford, $0 \in \underline{\mathrm{Conv}}_2(g_0 v_2)$ $(\subset \underline{\mathfrak{h}}^*(\mathbb{R}))$ $\forall g_0 \in \underline{G}_0(h)$. Note that since h acts on $V_2(h)$ by 2 & $\underline{\mathfrak{h}} = h^\perp$, we have

$$\mathrm{Conv}_2(u) = \chi + \underline{\mathrm{Conv}}_2(u) \ \forall \ u \in V_2 \setminus \{0\} \Rightarrow$$

(2) $$\chi \in \mathrm{Conv}_2(g_0 v_2) \ \forall g_0 \in G_0(h).$$

Now let $g \in G_{\geqslant 0}(h)$ & g_0 be its projection to $G_0(h)$. Then the projection of gv to $V_2(h)$ is $g_0 v_2$. Since $V_2(h)$ & $V_{>2}(h)$ are sums of weight spaces for T, we see $\mathrm{Supp}(g_0 v_2) \subset \mathrm{Supp}(gv)$. This and (2) prove the claim. $\square$

Step 3: Here we prove that h satisfying (1) is G-conjugate to any characteristic, h', of v. This will finish the proof of (6) $\Rightarrow$ (a).

We will need some more notation. Let $u \in V \setminus \{0\}$ be s.t.

$0 \notin \mathrm{Conv}(u)$. We write X_u for the point in $\mathrm{Conv}(u)$ closest to 0, $X_u \in \mathfrak{h}^*_{\mathbb{Q}}$. By Step 2, $X \in \mathrm{Conv}(gv) \subset \{x \mid \langle h, x\rangle \geq 2\}$ $\forall g \in G_{\geq 0}(h)$. Since X is the closest point to 0 in the ambient half-space, $X = X_{gv}$.

Set $h_u := \dfrac{2\iota^{-1}(X_u)}{(X_u, X_u)}$. This is the vector of minimal length among those that take values ≥ 2 on $\mathrm{Conv}(u)$ (exercise). By the construction of a characteristic in Sec 2 of Lec 12 we have $h' = h_{v'}$ where $v' = gv$ for some $g \in G$.

Consider the parabolic subgroups $G_{\geq 0}(h), G_{\geq 0}(h')$. They both contain Borel subgroups, B, B' containing T. The Bruhat decomposition tells us that $\exists!\, w \in W$ s.t. $g \in B'\dot{w}B$, where $\dot{w}$ is a lift of w to $N_G(T)$. In particular, $\exists\, p' \in G_{\geq 0}(h')$, $p \in G_{\geq 0}(h)$ s.t. $g = (p')^{-1}\dot{w}p$.

By Exercise in Sec 1.0, h' is a characteristic of $p'v'$ $\Rightarrow$ $h' = h_{p'v'}$. And $h = h_{pv}$ by a previous part of this step. So we can replace v, v' w. $pv, p'v'$ & assume that $g = \dot{w}$. Since $\dot{w}$ normalizes $\mathfrak{h}$, we have $\mathrm{Conv}(v') = w.\mathrm{Conv}(v)$ & since W acts on $\mathfrak{h}^*_{\mathbb{R}}$ by isometries, $X_{v'} = wX_v \Rightarrow h' = h_{v'} = wh_v = wh$ implying the claim.

Step 4: Here we prove part (2) of Thm. By Step 1, any characteristic of v satisfies (6), so by Step 2 it satisfies (1). It follows from Step 1 that $\exists\, g \in G \mid gh' = h$. It follows that h is a characteristic for both v, gv. Applying the last paragraph of Step 3, we see that $g=(p')^{-1}\dot{w}p$ w. $p,p'\in G_{\geqslant 0}(h)$ & $h=wh$. It follows that w is in the Weyl group of $G_0(h)$ hence $\dot{w}\in G_0(h) \Rightarrow g\in G_{\geqslant 0}(h)$ $\square$

2) Hesselink/ Kirwan-Ness stratification.

2.0) Definition of strata.

The notion of a characteristic allows to stratify $\pi^{-1}(\pi(0))$ into the union of smooth G-stable locally closed subvarieties.

Namely for an orbit $Gh\subset G.\mathfrak{h}_{\mathbb{Q}}$ we define a subset $V_{Gh}\subset$ $\subset\pi^{-1}(\pi(0))$ of all elements whose characteristic is in Gh. We want to describe V_{Gh}. In the notation of Sec 1 set $V_2(h)^0=V_2(h)\setminus \pi_2^{-1}(\pi_2(0))$ & let $V_{\geqslant 2}(h)^0$ denote the preimage of $V_2(h)^0$ in $V_{\geqslant 2}(h)$. By Theorem in Sec 1.1, we have $V_{\geqslant 2}(h)^0\subset V_{Gh}$. Also V_{Gh} is G-stable. So we get the action homomorphism whose image is in V_{Gh}.

$$\alpha: G\times V_{\geqslant 2}(h)^0 \longrightarrow V,\ (g,v)\mapsto gv.$$

Set $P:=G_{\geqslant 0}(h)$. Note that $V_{\geqslant 2}(h)^0$ is P-stable: $V_{\geqslant 2}(h)\subset V$ is $G_{\geqslant 0}(h)$-stable & $V_2(h)\subset V_{\geqslant 2}(h)$ is $G_0(h)$-stable by Sec 1.0. The subset $V_2(h)^0\subset V_2(h)$ is $G_0(h)$-stable & since $G_{\geqslant 0}(h)$ acts on $V_{\geqslant 2}(h)/V_{>2}(h)\xrightarrow{\sim} V_2(h)$ by the projection to $G_0(h)$, we see that $V_{\geqslant 2}(h)^0$ is P-stable.

So P acts on $G\times V_{\geqslant 2}(h)^0$ by $p.(g,v)=(gp^{-1}, pv)$. Note that α is P-invariant by construction.

The following proposition is a key ingredient in showing that V_{Gh} form a stratification (in a weaker sense: $\overline{V_{Gh}}$ is not a union of $V_{Gh'}$'s.

Proposition: $\mathrm{im}\,\alpha = V_{Gh}$. Each scheme-theoretic fiber of α is a single P-orbit.

Proof: By Thm in Sec 1.1, h is a characteristic of $v \Leftrightarrow v \in V_{\geq 2}(h)^0$. This implies $V_{Gh} = G V_{\geq 2}(h)^0 = \mathrm{im}\,\alpha$.

We now prove the claim about fibers on the level of subsets. It's enough to prove that for $u, v \in V_{\geq 2}(h)^0, g \in G$, the equality $u = gv \Rightarrow g \in P$. For this observe that $h, g.h$ are characteristics of u and use 2) of Thm.

To prove that the scheme-theoretic fibers are P-orbits (with their reduced scheme structures) it's enough to show that $\ker d_{(g,v)}\alpha = T_{(g,v)} P.(g,v)$. By G-equivariance we can assume that $g = 1$. Then $T_{(1,v)}(G \times V_{\geq 2}(h)^0) = \mathfrak{g} \oplus V_{\geq 2}(h)$ & $d\alpha_{(1,v)}(x,u) = x.v + u$. We need to show $x.v = -u \Rightarrow x \in \mathfrak{p}$.

Let us write $x = \sum_{a \in \mathbb{Q}} x_a$, $v = \sum_{b \geq 2} v_b$, where $x_a \in \mathfrak{g}_a(h)$, $v_b \in V_b(h)$. Choose min. $a \in \mathbb{Q}$ w. $x_a \neq 0$; $x \notin \mathfrak{p} \Leftrightarrow a < 0$. Then $x.v \in x_a . v_2 + \bigoplus_{b > a+2} V_b(h)$. So $x.v = -u \Rightarrow x_a . v_2 = 0 \Rightarrow x_a \in \mathrm{Lie}(\mathrm{Stab}_G(v_2))$. But $v_2 \in V_2(h)^0$ and so by Thm in Sec 1.1, h is a characteristic of v_2. Exercise in Sec 1.1 implies $\mathrm{Stab}_G(v_2) \subset P \Rightarrow x_a \in \mathfrak{p}$ leading to a contradiction.

Rem: In favorable situations, V_{Gh} is a single G-orbit. For example, consider $V=\mathfrak{g}$. We know, see Example in Sec. 1.1 that if (e,h,f) is an $\mathfrak{sl}_2$-triple, then h is a characteristic of e. By Malcev's thm (see [CM], Sec. 3.4) if $(e,h,f), (e',h,f')$ are $\mathfrak{sl}_2$-triples, then they are conjugate. It follows that each non-empty $\mathfrak{g}_{Gh}$ is a single nilpotent orbit.

2.1) Bonus: homogeneous bundles

It turns out that the previous proposition together with a construction in this section is sufficient to fully describe $V_{G\mathfrak{v}}$ & get some info re the closure.

A general construction of a homogeneous bundle is as follows. Let G be an algebraic group, H its algebraic (= Zariski closed) subgroup & Y be a quasi-projective variety with an H-action. Then $G\times H$ acts on $G\times Y$ via:

$$(g,h).(g',y)=(gg'h^{-1},hy)$$

It turns out that there is a variety $G\times^H Y$ with the following properties (see [PV], Sec. 4.8) & their easy corollaries.

1) It comes with a morphism $G\times Y\longrightarrow G\times^H Y$ that is a principal H-bundle in etale topology, i.e. $G\times^H Y$ is a quotient of $G\times Y$ by H in the strongest sense.

2) $G \curvearrowright G\times^H Y$ uniquely so that $G\times Y \to G\times^H Y$ is G-equivariant.

3) $G\times^H Y \longrightarrow G/H$, $H.(g,y)\mapsto gH$, is locally trivial (in etale topology) with fiber Y over $1H\subset G/H$.

4) The construction is functorial in Y: if $\varphi: Y\to Y'$ is an H-equivariant morphism, then $\tilde{\varphi}: G\times^H Y \longrightarrow G\times^H Y'$, $H.(g,y)\mapsto H.(g,\varphi(y))$ is a G-equivariant morphism. If φ is a closed (resp. open) embedding, then so is $\tilde{\varphi}$.

5) If the action of H on Y extends to an action of G, then $G\times^H Y \xrightarrow{\sim} G/H\times Y$ via $H.(g,y)\mapsto (gH, gy)$.

In particular, we can apply this construction to connected reductive G, $H=P$ & $Y=V_{\geq 2}(h)^0$. Since Y is smooth, 1) (or 3)) implies that $G\times^P V_{\geq 2}(h)^0$ is smooth. Now we have a morphism

$$\underline{\alpha}: G\times^P V_{\geq 2}(h)^0 \longrightarrow V,\quad P.(g,v)\mapsto gv.$$

Proposition means that the scheme-theoretic fibers of $\underline{\alpha}$ are points. So $\underline{\alpha}$ is a locally closed embedding. And the image is V_{Gh}, showing it's a locally closed smooth subvariety.

We can also study $\overline{V_{Gh}}$ using this construction. Namely, we get an open inclusion $G\times^P V_{\geq 2}(h)^0 \hookrightarrow G\times^P V_{\geq 2}(h)$ & $\underline{\alpha}$ factors as this inclusion followed by $\underline{\tilde{\alpha}}: G\times^P V_{\geq 2}(h) \longrightarrow V$

Lemma: $\underline{\tilde{\alpha}}$ is projective.

Proof: $\underline{\tilde{\alpha}}$ factors as $G\times^P V_{\geq 2}(h) \hookrightarrow G\times^P V \xrightarrow{\sim} G/P\times V \xrightarrow{pr_2} V$

4) 5) projection, projective! $\square$

Exercises: 1) $\mathrm{im}\,\underline{\tilde{\alpha}} = \overline{V_{Gh}}$ & $\underline{\tilde{\alpha}}: G\times^P V_{\geq 2}(h) \to \overline{V}_{Gh}$ is a resolution of singularities.

2) $\overline{V_{Gh}} \setminus V_{Gh} \subset \coprod_{h' \mid (h',h') < (h,h)} V_{Gh'}$.