Lecture 14, 2/26/25

1) Luna slice theorem

Ref: [PV], Secs 6.1-6.7.

1) Luna slice theorem

1.0) Setup

The base field is $\mathbb{C}$.

The goal of this lecture is to answer the following:

Question: Let G be a reductive group acting on an affine variety X, and let $x \in X$ be a point with closed G-orbit. We want to understand the structure of the action "near Gx" & the structure of $X/\!/G$ near $\pi(x)$.

An important ingredient in the answer is the subgroup $H := \mathrm{Stab}_G(x)$. This group is reductive for the following reason: Gx is closed in X hence is affine. And $Gx = G/H$ as a variety.

The following will be proved in the next lecture (non-algebraically) independently of this one.

Fact: Let $H \subset G$ be an algebraic subgroup. If G/H is affine, then H is reductive.

1.1) Homogeneous bundle

Here we will explain a framework for answering Question in Section 1. For this we will need a special case of the construction sketched in Sec 2.1 of Lec 13.

Let H be a reductive subgroup of G & S be a finite type affine scheme with an H-action. Consider the scheme $G\times S$. It comes with an action of $G\times H$:

$$(g,h).(g',s):=(gg'h^{-1}, hs).$$

We write $G\times^H S:=(G\times S)/\!/H$. The group G acts on $G\times^H S$ b/c actions of G & H on $G\times S$ commute. The morphism $G\times S\to G$ induces $p: G\times^H S=(G\times S)/\!/H \longrightarrow G/\!/H=G/H$ s.t. the following is commutative:

$$\begin{tikzcd}
G\times S \arrow[r, "pr_1"] \arrow[d, "\pi"] & G \arrow[d, "\pi"] \\
G\times^H S \arrow[r, dashed, "p"] & G/H
\end{tikzcd}$$

Exercise: 1) Show that p is G-equivariant

2) Identify $p^{-1}(1H)$ w. S (hint: $p^{-1}(1H)\simeq[(p\circ\pi)^{-1}(1H)]/\!/H$). In particular, $\dim G\times^H S=\dim G/H+\dim S$.

3) Establish an isomorphism $(G\times^H S)/\!/G\xrightarrow{\sim} S/\!/H$ (hint: compute $\mathbb{C}[G\times S]^{G\times H}$ in 2 different ways).

Thx to 1) & 2), $G\times^H S\longrightarrow G/H$ can be viewed as a bundle over G/H w. fiber S over $1H$. This & 1) justify the name "homogeneous bundle" for $G\times^H S$. For more on these, see [PV], Sec 4.8.

Rem: One can view the construction of $G\times^H S$ as an induction that starts from a variety w. H-action and produces a variety w. a G-action. Essentially any property of the G-action on $G\times^H S$ can be recovered from the H-action on S, cf. 3) of Exercise.

1.2) Slice

Let's return to the setup of Sec 1.0. In particular, $H=\mathrm{Stab}_G(x)$. We are going to compare X with $G\times^H S$ for a suitable locally closed H-stable affine subvariety $S\subset X$, a "slice".

First, note that in this generality there's a G-equivariant morphism $G\times^H S\longrightarrow X$: the action map $G\times S\xrightarrow{\alpha} X$, $(g,s)\mapsto gs$ is H-invariant & so uniquely descends to $\alpha': G\times^H S\to X$: $H.(g,s)\mapsto gs$.

Now we need to explain how to choose S. It's going to be open in a closed H-stable subscheme $\overline{S}\subset X$. The construction of $\overline{S}$ is in two steps:

I) Assume $X=V$, a rational G-representation. Since H is reductive, one can find an H-stable subspace $N\subset V$ w. $N\oplus \mathfrak{g}.x=V$. As an H-representation, $N\simeq V/(\mathfrak{g}/\mathfrak{h})$, a normal space to $Gx\subset V$. Knowing $\mathfrak{h}$ recovers N completely (again, b/c the representation of H in V is completely reducible). Then we set $N_x:=x+N$ and take $S:=N_x$.

II) In the general case, we can G-equivariantly embed X into some rational representation, V (Step 1 in Sec. 1.1.4 of Lec 11). Let $N\subset V$ be as above. Then we set

$$\overline{S}:=N_x\cap X \text{ (a scheme theoretic intersection).}$$

Lemma:

1) We have $G\times^H\overline{S}\xrightarrow{\sim}X\times_V(G\times^H N_x)$.

2) The morphism $G\times^H\overline{S}\longrightarrow X$ is etale at $H.(1,x)$.

Proof:

1) Note that $\overline{S}=X\cap N_x \Rightarrow G\times\overline{S}$ is the preimage of X in $G\times N_x$ $\Leftrightarrow G\times\overline{S}\xrightarrow{\sim}X\times_V(G\times N_x)$. We want to show that $[X\times_V(G\times N_x)]/\!/H\xrightarrow{\sim}X\times_V([G\times N_x]/\!/H)$. This follows from:

Exercise: Let $\widetilde{Y},Y,X$ be finite type affine schemes w. morphisms $\widetilde{Y}\to Y$ & $X\to Y$. Let H be a reductive group acting on $\widetilde{Y}$ s.t.

$\tilde{Y}\to Y$ is invariant. Then $(X\times_Y \tilde{Y})/\!/H \xrightarrow{\sim} X\times_Y(\tilde{Y}/\!/H)$

Hint: Use that $\mathbb{C}[\tilde{Y}]^H$ is naturally a direct summand in $\mathbb{C}[\tilde{Y}]$ to show that

$$(*)\qquad \mathbb{C}[X]\otimes_{\mathbb{C}[Y]}\mathbb{C}[\tilde{Y}]^H \longrightarrow \left(\mathbb{C}[X]\otimes_{\mathbb{C}[Y]}\mathbb{C}[\tilde{Y}]\right)^H$$

is injective & $\left(\mathbb{C}[X]\otimes_{\mathbb{C}}\mathbb{C}[\tilde{Y}]\right)^H \twoheadrightarrow \left(\mathbb{C}[X]\otimes_{\mathbb{C}[Y]}\mathbb{C}[\tilde{Y}]\right)^H$ to show that $(*)$ is surjective.

2): Since being etale is stable under base change, we reduce to showing that $G\times^H N_x \longrightarrow V$ is etale at $H.(1,x)$. $\dim G\times^H N_x = \dim G+\dim N-\dim H=\dim V$ so it's enough to show that $d_{H.(1,x)}\alpha'$ is surjective. Thx to the commutative diagram

$$\begin{tikzcd}
G\times S \arrow[d] \arrow[drr, "\alpha"] & & \\
G\times^H S \arrow[rr, "\alpha'"] & & X
\end{tikzcd}$$

it suffices to check that $d_{(1,x)}\alpha$ is surjective. The latter is because $\operatorname{im} d_{(1,x)}\alpha \supset T_x Gx + T_x N_x = \mathfrak{g}.x + N = V$. $\square$

1.3) Excellent morphisms.

Let G be a reductive group and X, Y be finite type affine schemes. Let G act on X & Y & $\varphi: Y\to X$ be a G-equivariant morphism. Let $\underline{\varphi}: Y/\!/G \longrightarrow X/\!/G$ be the induced morphism of quotients. The following definition is due to Luna. It describes properties of $\underline{\alpha}: G\times^H S \to X$ we want to have:

Definition: We say that φ is *excellent* if

(a) $\underline{\varphi}: Y/\!/G \to X/\!/G$ is etale and

(b) The following is Cartesian:

$$\begin{tikzcd} Y \arrow[r, "\varphi"] \arrow[d, "\pi_Y"'] & X \arrow[d, "\pi_X"] \\ Y/\!/G \arrow[r, "\underline{\varphi}"] & X/\!/G \end{tikzcd}$$

$\leftarrow$ quotient morphisms.

Properties of excellent morphisms: Suppose φ is excellent. Then

1) φ is etale.

2) $\forall\, y\in Y$, φ restricts to $\pi_Y^{-1}(\pi_Y(y)) \xrightarrow{\sim} \pi_X^{-1}(\pi_X(\varphi(y)))$.

3) Let $X_0\subset X$ be a closed G-stable subscheme & $Y_0 = X_0\times_X Y$. If $Y\to X$ is excellent, then so is $Y_0\to X_0$.

Proof:

1) & 2) are exercises. To prove 3) observe that $Y_0 = X_0\times_X Y$ & $Y\xrightarrow{\sim} X\times_{X/\!/G} Y/\!/G$ yields $Y_0 \xrightarrow{\sim} X_0\times_{X/\!/G} Y/\!/G$, which by Exercise in Sec. 1.2, implies $Y_0/\!/G \xrightarrow{\sim} X_0/\!/G\times_{X/\!/G} Y/\!/G$. So the morphism $Y_0/\!/G\to X_0/\!/G$ is etale as a base change of $Y/\!/G\to X/\!/G$; $Y_0 \xrightarrow{\sim} X_0\times_{X/\!/G} Y/\!/G \xrightarrow{\sim} X_0\times_{X_0/\!/G}(X_0/\!/G\times_{X/\!/G} Y/\!/G) \xrightarrow{\sim} X_0\times_{X_0/\!/G} Y_0/\!/G$ establishing the two conditions of an excellent morphism $\square$

The following is the main technical result that will be proved in a separate note.

Main Lemma: Suppose X, Y are, in addition, smooth. Let $y \in Y$ be s.t. Gy, $G\varphi(y)$ are closed. Moreover, suppose:

I) φ is etale at y, and

II) $\varphi: Gy \xrightarrow{\sim} G\varphi(y)$.

Then $\exists$ open affine $(Y/\!/G)^\circ \subset Y/\!/G$ containing $\pi_Y(y)$ s.t. the restriction $\varphi: \pi_Y^{-1}((Y/\!/G)^\circ) \longrightarrow X$ is excellent.

1.4) Etale slice theorem.

Definition: Let $x \in X$ be a point with a closed G-orbit, and $H := \mathrm{Stab}_G(x)$. By an etale slice at x we mean an H-stable locally closed affine subscheme $S \subset X$ containing x s.t. the morphism $G \times^H S \longrightarrow X$ is excellent.

Thm (Luna): An etale slice at x exists.

Proof (modulo Main Lemma)

Thx to property 3) in Sec 1.3 & 1) of Lemma in Sec 1.2 we can reduce to the case when $X = V$, $\bar{S} = N_x$ (the details of this reduction are left as an exercise). The conditions of Main Lemma are satisfied: to see that $G\times^H N_x$ is smooth notice that it is a bundle over smooth G/H with smooth fibers, N_x, see Exercise in Sec. 1.1. For S we take the preimage of $(G\times^H \bar{S})^\circ /\!/ G \subset (G\times^H \bar{S})/\!/G = \bar{S}/\!/H$ under $\bar{S} \twoheadrightarrow \bar{S}/\!/H$. $\square$

Corollary: Suppose G acts on X freely. Then $\pi: X \to X/\!/G$ is a locally trivial (in etale topology) G-bundle.

Proof: exercise.

1.5) Case of smooth X.

Suppose, in addition, that X is smooth. Then S is smooth: this is because both $G\times S \xrightarrow{\pi_H} G\times^H S \xrightarrow{\alpha'} X$ are smooth morphisms: α' is etale by Luna's thm & π_H is smooth by Corollary, both in Sec 1.4.

Let $\mathfrak{m}\subset \mathbb{C}[S]$ be the max'l ideal of x. Set $U := T_x S = (\mathfrak{m}/\mathfrak{m}^2)^*$. The projection $\mathfrak{m} \twoheadrightarrow U^*$ is H-equivariant and, by complete reducibility admits an H-equivariant section $U^* \to \mathfrak{m}$ yielding an H-equivariant morphism $S \xrightarrow{\varphi} U$ w. $x\mapsto 0$. By the construction φ is etale at x. So we can find an open affine $(U/\!/H)^\circ \subset U/\!/H$ containing 0 and replace S with $(U/\!/H)^\circ \times_{S/\!/H} S$ to achieve that φ is excellent. We get the following commutative diagram, where the squares are Cartesian and horizontal arrows are etale.

$$\begin{tikzcd} G\times^H U \arrow[d] & G\times^H S \arrow[l] \arrow[r] \arrow[d] & X \arrow[d] \\ U/\!/H & S/\!/H \arrow[l] \arrow[r] & X/\!/G \end{tikzcd}$$

This reduces the study of X near Gx to the study of the linear action of H on U. Note that $U = T_x S \simeq T_x X / T_x Gx$.

Remark: In particular we see that etale locally near $\pi(x)$ $X/\!/G$ behaves like $U/\!/H$.

This observation gives an inductive tool to investigate good properties of quotients for linear actions. Let $X=V$ be a vector space. If $V/\!/G$ is smooth, then $U/\!/H$ is smooth as well. Details are left as an exercise.

1.6) Some corollaries.

Lemma 1: Let X be an affine variety equipped with an action of a reductive group G. Let $x\in X$ be a point w. a closed G-orbit. Let $H = \mathrm{Stab}_G(x)$. Then $\exists$ a G-stable open subset $X^\circ \subset X$ containing x s.t. $\forall y \in X^\circ$ the stabilizer of y is conjugate in G to a subgroup of H.

Proof: Let S be etale slice. Take X° to be the image of $G\times^H S$ in X. This is open b/c the morphism $\varphi: G\times^H S \to X$ is etale. Note that $G\times^H S \xrightarrow{\sim} S/\!/H \times_{X/\!/G} X$. For $(s,x)\in S/\!/H\times_{X/\!/G} X$, the stabilizer in G coincides w. $\mathrm{Stab}_G(x)$. On the other hand, thx to the epimorphism $G\times^H S \xrightarrow{p} G/H$ every stabilizer in $G\times^H S$ is conjugate to a subgroup of H. So every stabilizer in X°, the image of the projection $S/\!/H\times_{X/\!/G} X \to X$, is conjugate to a subgroup of H $\square$

Lemma 2: Let G be a reductive group acting on a smooth affine variety X. Then the fixed point locus X^G is smooth.

Proof:

Let $x\in X^G$. Apply the construction from Sec. 1.6. We get an etale morphism $(X/\!/G)^\circ \longrightarrow V/\!/G$ for some open $(X/\!/G)^\circ$ containing $\pi(x)$ & a G-equivariant isomorphism $\pi^{-1}((X/\!/G)^\circ) \xrightarrow{\sim} (X/\!/G)^\circ \times_{V/\!/G} V$. Then $\pi^{-1}((X/\!/G)^\circ)^G \xrightarrow{\sim} (X/\!/G)^\circ \times_{V/\!/G} V^G$. Of course V^G is a vector subspace, hence is smooth. Since $(X/\!/G)^\circ \to V/\!/G$ is etale, $\pi^{-1}((X/\!/G)^\circ)^G$ is also smooth finishing the proof. $\square$

Exercise: For $x\in X^G$, we have $T_x(X^G) = (T_x X)^G$.

Rem: If G is not reductive, the claim may fail: consider the action of $(\mathbb{C},+)$ on $\mathbb{C}^3$ by $\alpha.(x,y,z) := (x,y,z+\alpha f(x,y))$, where $f\in\mathbb{C}[x,y]$ is a polynomial. The fixed point locus is

$$\{(x,y) \mid f(x,y)=0\}\times\mathbb{A}^1.$$