Proof of Main Lemma, Sec 1.3 of Lec 14

Recall that the lemma is follows:

Main Lemma: Suppose X, Y are irreducible & smooth. Let $y \in Y$ be s.t. Gy, $G\varphi(y)$ are closed. Moreover, suppose:

I) φ is etale at y, and

II) $\varphi: Gy \xrightarrow{\sim} G\varphi(y)$.

Then $\exists$ open affine $(Y/\!/G)^\circ \subset Y/\!/G$ containing $\pi_Y(y)$ s.t. the restriction $\varphi: \pi_Y^{-1}((Y/\!/G)^\circ) \longrightarrow X$ is excellent, i.e.

(a) $\underline{\varphi}: (Y/\!/G)^\circ \longrightarrow X/\!/G$ is etale

(b) $Y^\circ \xrightarrow{\sim} (Y/\!/G)^\circ \times_{X/\!/G} X$, where $Y^\circ = \pi_Y^{-1}((Y/\!/G)^\circ)$.

Notation: For a finite type affine scheme $\tilde{Z}$ & a closed subscheme Z, we consider the completion $\mathbb{C}[\tilde{Z}]^{\wedge Z} := \varprojlim_{n\to\infty} \mathbb{C}[\tilde{Z}]/I_Z^n$ where I_Z is the defining ideal of Z. Relevant completions include: $\mathbb{C}[X]^{\wedge x}$, $\mathbb{C}[Y]^{\wedge y}$, $\mathbb{C}[X]^{\wedge Gx}$, $\mathbb{C}[Y]^{\wedge Gy}$, $\mathbb{C}[X/\!/G]^{\wedge x}$ (we abuse the notation and write x instead of $\pi_X(x)$), $\mathbb{C}[Y/\!/G]^{\wedge y}$.

The proof is in 3 steps:

1) Prove that $\varphi^*: \mathbb{C}[X]^{\wedge Gx} \xrightarrow{\sim} \mathbb{C}[Y]^{\wedge Gy}$.

2) Prove that $\pi_X^*: \mathbb{C}[X/\!/G]^{\wedge x} \xrightarrow{\sim} (\mathbb{C}[X]^{\wedge Gx})^G$, and the same for Y. This together w. 1) implies $\underline{\varphi}$ is etale at $\pi_Y(y)$.

3) Finish the proof.

1) Step 1: $\varphi^*: \mathbb{C}[X]^{\wedge_{Gx}} \xrightarrow{\sim} \mathbb{C}[Y]^{\wedge_{Gy}}$.

Note that the composition $\mathbb{C}[X] \xrightarrow{\varphi^*} \mathbb{C}[Y] \longrightarrow \mathbb{C}[Y]^{\wedge_{Gy}}$ extends to $\mathbb{C}[X]^{\wedge_{Gx}} \longrightarrow \mathbb{C}[Y]^{\wedge_{Gy}}$ b/c $\varphi^*(I_{Gx}) \subset I_{Gy} \Rightarrow \varphi^*(I_{Gx}^k) \subset I_{Gy}^k$ $\forall k \geqslant 0$. Denote the extension also by φ^*.

We note that the algebras $\mathbb{C}[X]^{\wedge_{Gx}}$, $\mathbb{C}[Y]^{\wedge_{Gy}}$ come w. descending filtrations (as inverse limits) & φ^* is a filtered algebra homomorphism. The filtrations are complete & separated so to show φ^* is iso it is enough to show that $\operatorname{gr}\varphi^*: \operatorname{gr}\mathbb{C}[X]^{\wedge_{Gx}} \xrightarrow{\sim} \operatorname{gr}\mathbb{C}[Y]^{\wedge_{Gy}}$. Note that $\operatorname{gr}\mathbb{C}[X]^{\wedge_{Gx}} = \bigoplus_{i=0}^{\infty} I_{Gx}^i / I_{Gx}^{i+1} = \mathbb{C}[N_X(Gx)]$ (a normal bundle). Since φ is G-equivariant & etale at y, it's etale at every point of Gy. And since it restricts to $Gy \xrightarrow{\sim} Gx$, it gives rise to an isomorphism $N_Y(Gy) \xrightarrow{\sim} N_X(Gx)$. The corresponding pullback homomorphism is nothing else but $\operatorname{gr}\varphi^*$ finishing the proof.

2) Step 2: $\pi_X^*: \mathbb{C}[X/\!/G]^{\wedge_x} \xrightarrow{\sim} (\mathbb{C}[X]^{\wedge_{Gx}})^G$

We will prove a more general claim:

Lemma: Let A be a finitely generated commutative algebra equipped with a rational representation of a reductive group G by algebra automorphisms. Let $I \subset A$ be a G-stable ideal.

Set $\underline{A} := A^G$, $\underline{I} := I^G$. Then

$$\varprojlim_n \underline{A}/\underline{I}^n \xrightarrow{\sim} (\varprojlim_n A/I^n)^G.$$

Proof:

Note that $\underline{I}^n \subset (I^n)^G$ so we indeed have a homomorphism $\varprojlim \underline{A}/\underline{I}^n \longrightarrow (\varprojlim A/I^n)^G$. To show that this is an isomorphism we need to show that the filtrations $\underline{I}^n$, $(I^n)^G$ on $\underline{A}$ are compatible which reduces to:

$(*)$ $\forall n \geq 0\ \exists\ m \geq 0 \mid (I^m)^G \subset \underline{I}^n$.

This is done by using the usual trick with blow-up algebras: set $Bl_I(A) := \bigoplus_{n\geq 0} I^n$. This is a finitely generated algebra w. rational action of G, so $Bl_I(A)^G = \bigoplus_{n\geq 0} (I^n)^G$ is finitely generated. Choose a finite collection of homogeneous generators $f_1, \dots f_k$ of the $\underline{A}$-algebra $Bl_I(A)^G$, of degrees $d_1, \dots d_k > 0$. By the construction $(I^n)^G \subset Span_{\underline{A}}(f_1^{a_1} \dots f_k^{a_k} \mid \sum a_i d_i = n) \subset [f_i \in I^G = \underline{I}$; set $d := \max(d_i)]$ $\underline{I}^{\lceil n/d \rceil}$. This proves $(*)$. $\square$

To apply this to our situation, observe that the maximal ideal of $\pi_X(x)$ in $\mathbb{C}[X/\!/G]$ is I_{Gx}^G (exercise). So we have isomorphisms:

$$\mathbb{C}[X/\!/G]^{\wedge x} \xrightarrow{\sim} (\mathbb{C}[X]^{\wedge Gx})^G \xrightarrow{\sim} (\mathbb{C}[Y]^{\wedge Gy})^G \xleftarrow{\sim} \mathbb{C}[Y/\!/G]^{\wedge y}$$

The resulting isomorphism $\mathbb{C}[X/\!/G]^{\wedge x} \xrightarrow{\sim} \mathbb{C}[Y/\!/G]^{\wedge y}$ is nothing but φ^*.

Hence φ is etale at $\pi_Y(y)$.

3) Finishing the proof

3.1) First of all, the locus in $Y/\!\!/G$, where $\varphi: Y/\!\!/G \to X/\!\!/G$ is etale, is open (this is the locus where the sheaf of relative Kähler differentials is locally free of rk 1). So we can find $f_1 \in \mathbb{C}[Y]^G$ s.t. $f_1(y) \neq 0$ & φ is etale on $(Y/\!\!/G)_{f_1}$. We can then replace Y w. Y_{f_1} (so that $Y_{f_1}/\!\!/G \xrightarrow{\sim} (Y/\!\!/G)_{f_1}$) and assume φ is etale.

Set $Y' = Y/\!\!/G \times_{X/\!\!/G} X$; Y' is smooth b/c X is smooth. What remains to do is to find $f_2 \in \mathbb{C}[Y]^G = \mathbb{C}[Y']^G$ s.t. $f_2(y) \neq 0$ & $Y_{f_2} \xrightarrow{\sim} Y'_{f_2}$. We write y' for the image of y in Y' so that $Gy \xrightarrow{\sim} Gy'$.

3.2) First, we claim that $\exists f_3 \in \mathbb{C}[Y]^G$ s.t. $Y_{f_3} \to Y'_{f_3}$ is etale, $f_3(y) \neq 0$. Let $Z \subset Y$ be the locus, where $Y \to Y'$ fails to be etale. It's closed & G-stable. It doesn't contain y. Since Gy is the only closed G-orbit in $\pi_Y^{-1}(\pi_Y(y))$, the closed G-stable subvariety $Z \cap \pi_Y^{-1}(\pi_Y(y))$ must be empty. So the closed subvariety $\pi_Y(Z) \subset Y/\!\!/G$ doesn't contain $\pi_Y(y)$. Pick f_3 vanishing on $\pi_Y(Z)$ but not at $\pi_Y(y)$. Replacing Y w. Y_{f_3} we achieve that $Y \to Y'$ is etale.

3.3) We claim that can find $f_4 \in \mathbb{C}[Y]^G$ s.t. $Y_{f_4} \to Y'_{f_4}$ is finite. Since $Y \to Y'$ is etale, it's quasi-finite (= all fibers are finite). Note that by the Zariski main thm for quasi-finite morphisms, we can factorize $Y \to Y'$ as $Y \overset{\iota}{\hookrightarrow} \bar{Y} \overset{\psi}{\twoheadrightarrow} Y'$, where $Y \overset{\iota}{\hookrightarrow} \bar{Y}$ is an open embedding & $\bar{Y} \overset{\psi}{\twoheadrightarrow} Y'$ is finite. Note that $\mathbb{C}[\bar{Y}]$ is defined as the integral closure of $\mathbb{C}[Y']$ in $\mathbb{C}[Y]$. In particular, it is G-stable. Since $\mathbb{C}[Y'] \overset{\psi^*}{\hookrightarrow} \mathbb{C}[\bar{Y}] \overset{\iota^*}{\hookrightarrow} \mathbb{C}[Y]$, we have $\mathbb{C}[Y']^G \hookrightarrow \mathbb{C}[\bar{Y}]^G \hookrightarrow \mathbb{C}[Y]^G$. Since $\mathbb{C}[Y]^G = \mathbb{C}[Y']^G$, this implies $\mathbb{C}[\bar{Y}]^G = \mathbb{C}[Y]^G$.

Observe that Gy is closed in $\bar{Y}$. Indeed, $\psi(\overline{Gy})$ is the closure of $\psi(Gy) = Gy'$ but Gy' is closed. It follows that $\forall$ orbit in $\overline{Gy}$ goes to Gy' but for dimension reasons, only Gy can.

Now we argue as in 3.2) to show that $\pi_{\bar{Y}}(\bar{Y} \setminus Y)$ is a closed subvariety in $\bar{Y}/\!/G = Y/\!/G$ that doesn't contain $\pi_Y(y)$. So we can find $f_4 \in \mathbb{C}[Y]^G$ w. f_4 vanishing on $\pi_Y(\bar{Y} \setminus Y)$ but not at $\pi_Y(y)$. Then $Y_{f_4} \to Y'_{f_4}$ is etale and finite.

3.4) We now prove that $Y \xrightarrow{\sim} Y'$ (no need to cut further). Let d denote the degree of this morphism. We need to prove that $d = 1$. Let $y = y_1, \dots y_d$ be the preimages of $y' \in Y'$. Arguing as in 3.3), we see that Gy_i are closed. But then $\pi_Y(y_i) = \pi_{Y'}(y')$.

Since every fiber of the quotient morphism π_Y contains a unique closed orbit, we see that $Gy_i = Gy$ $\forall i$. And since $Gy \xrightarrow{\sim} Gy'$, we finally get $d = 1$.