Lecture 15, 03/03/25

1) Kempf-Ness theorem

2) Applications.

Refs: [PV], Sec. 6.12; [CdS], Part VIII; [L].

1) Kempf-Ness theorem

1.1) Statement

Let G be a reductive group & V be its finite dimensional rational representation. Let $K\subset G$ be a maximal compact subgroup of G, it's defined up to G-conjugation. There is a K-invariant Hermitian scalar product $(\cdot,\cdot)$ on V. Note that $(xv,v)\in\sqrt{-1}\mathbb{R}\ \forall x\in\mathfrak{k}\ \&\ v\in V$.
Define a map $\mu: V\longrightarrow \mathfrak{k}^*$ by $\langle\mu(v),x\rangle=\sqrt{-1}(xv,v)$

Exercise: μ is K-equivariant.

Remark: Here's how one computes $\langle\mu(v),x\rangle, x\in\mathfrak{k}$. The element x is skew-Hermitian so $V=\bigoplus_{\alpha\in\mathbb{R}} V_\alpha$, where x acts on V_α by $\sqrt{-1}\alpha$.
Write $v=\sum_\alpha v_\alpha$ w. $v_\alpha\in V_\alpha$. Then

$$\langle\mu(v),x\rangle=\sqrt{-1}(xv,v)=[(V_\alpha,V_\beta)=0 \text{ for } \alpha\neq\beta]=-\sum_\alpha \alpha(v_\alpha,v_\alpha)$$

Here's the main result for this lecture.

Theorem (Kempf-Ness) Let $v \in V$.

1) $Gv \cap \mu^{-1}(0) \neq \emptyset \Leftrightarrow Gv$ is closed.

2) If $Gv \cap \mu^{-1}(0) \neq \emptyset$, then this intersection is a single K-orbit.

1.2) Preparation

We need some of the ingredients from Lec 11:

I) Cartan decomposition: we have a maximal torus $T \subset G$ w. $T_K := K \cap T$ being maximal compact in T. Then we have $G = KTK$. Moreover, $T = T_K \times \exp(\mathfrak{h}_{\mathbb{R}})$ (where $\mathfrak{h} = Lie(T)$, $\mathfrak{h}_{\mathbb{R}} = \mathbb{R} \otimes_{\mathbb{Z}} \mathfrak{X}_*(T)$; under identification $T \simeq (\mathbb{C}^\times)^n$, we have $\exp(\mathfrak{h}_{\mathbb{R}}) = (\mathbb{R}_{>0})^n$). Then we have

$$G = K \exp(\mathfrak{h}_{\mathbb{R}}) K.$$

II) The following was established in proving the Hilbert-Mumford thm: if $u, v \in V$ are s.t. $Gu \subset \overline{Gv}$, then $\exists\, \kappa \in K$ s.t. $\overline{T\kappa v} \cap Gu \neq \emptyset$ (Sec. 1.2.2 of Lec 11). Moreover, by Sec 1.1.4 in Lec 11, $\exists\, \gamma \in \mathfrak{X}_*(T)$ s.t. $\lim_{t \to 0} \gamma(t) \kappa v$ exists and lies in Gu.

1.3) Proof of Thm.

Step 1: Gv is closed $\Rightarrow Gv \cap \mu^{-1}(0) \neq \emptyset$: Since Gv is closed $\exists$ point $v_0 \in Gv$ with minimal (v_0, v_0). Take $\xi \in \mathfrak{k}$ and consider

$$v_t = \exp(t\sqrt{-1}\,\xi) v_0, \quad t \in \mathbb{R}$$

Note that $\frac{d}{dt}(v_t, v_t)\big|_{t=0} = (\sqrt{-1}\,\xi v_0, v_0) + (v_0, \sqrt{-1}\,\xi v_0) = [\sqrt{-1}\,\xi$

acts by Hermitian operator$] = 2\sqrt{-1}(\xi v_0, v_0)$. But $t=0$ is a point of minimum of $t \mapsto (v_t, v_t)$. Hence $\sqrt{-1}(\xi v_0, v_0) = \mu(v_0) = 0$.

Step 2: $Gv \cap \mu^{-1}(0) \neq \emptyset \Rightarrow Gv$ is closed: Can assume $\mu(v)=0$. By II) in Sec 1.2, can replace v w. kv (this doesn't violate $\mu(v)=0$ by Exercise in Sec. 1.1) and assume $\lim_{t\to 0} \gamma(t) v \in Gu$ (for the unique closed $Gu \subset \overline{Gv}$).
Let $x = \sqrt{-1}\, d_1\gamma(1) \in \sqrt{-1}\mathfrak{h}^{k}_{\mathbb{R}} = Lie(T_k)$. We have decomposition

$$V = \bigoplus_{n\in\mathbb{Z}} V_n, \quad V_n = \{v \in V \mid \gamma(t) v = t^n v \Leftrightarrow xv = \sqrt{-1} n v\}$$

Then existence of $\lim_{t\to 0} \gamma(t) v$ is equivalent to $v_i = 0$ for $i<0$.
On the other hand, by Remark in Sec 1.1,

$$\langle \mu(v), x\rangle = -\sum_{n\in\mathbb{Z}} n (v_n, v_n).$$

Combining $\langle\mu(v), x\rangle = 0$ w. $v_i = 0\ \forall i<0$, we arrive at $v = v_0 \Rightarrow \lim_{t\to 0} \gamma(t) v = v \Rightarrow Gu = Gv \Rightarrow Gv$ is closed.

Step 3: We prove 2). It remains to prove that if $v \in \mu^{-1}(0)$ & $g\in G$ are s.t. $\mu(gv) = 0$, then $gv \in Kv$. Write g as $k_1 \exp(x) k_2$ for $k_i \in K$, $x \in \mathfrak{h}_{\mathbb{R}}$, see I) in Sec. 1.2. Replacing v w. $k_2 v$ we can assume $\mu(\exp(x)v) = 0$.
Write v as $\sum_{a\in\mathbb{R}} v_a$ w. $x.v_a = a v_a \Rightarrow \exp(x) v_a = e^a v_a$. Then

$\mu(v) = 0 \Leftrightarrow \sum a (v_a, v_a) = 0 \Leftrightarrow$

(1) $$\sum_{a>0} a(v_a, v_a) = \sum_{b<0} (-b)(v_b, v_b)$$

$\mu(\exp(x) v) = 0 \Leftrightarrow \sum a e^{2a} (v_a, v_a) = 0 \Leftrightarrow$

(2) $$\sum_{a>0} a\underbrace{e^{2a}}_{>1}(v_a,v_a) = \sum_{b<0}(-b)\underbrace{e^{2b}}_{<1}(v_b,v_b)$$

It follows that l.h.s of (2) $\geqslant$ l.h.s. of (1) w. equality iff $(v_a,v_a)=0$ $\forall a>0$, while r.h.s. of (2) $\leqslant$ r.h.s. of (1) w. equality iff $(v_b,v_b)=0$ $\forall b<0$. It follows that $\mu(\exp(x)v)=0 \Rightarrow \exp(x)v=v$ finishing the proof. $\square$

2) Applications

Our main goal here is to prove the following result already mentioned in Lec 14, due to Matsushima & Onishchik:

Thm: Let G be a reductive group & $H\subset G$ an algebraic subgroup. If G/H is affine, then H is reductive.

The proof is based on the Kempf-Ness theorem as well as understanding the nature of the map μ: it's a moment map.

2.1) Moment maps.

Let M be a C^∞-manifold. Recall that by a symplectic form on M one means a closed & non-degenerate 2-form ω. A basic example is a non-degenerate skew-symmetric $\mathbb{R}$-bilinear form on a real vector space. In particular, if V is a $\mathbb{C}$-vector space w.

Hermitian scalar product $(\cdot;\cdot)$, then $\omega = -2\cdot \mathrm{Im}(\cdot;\cdot)$ is a symplectic form (on the real vector space V).

Now suppose K is a Lie group acting on M preserving ω. The action induces a Lie algebra homomorphism $\mathfrak{k} \to \mathrm{Vect}(M)$, $x \mapsto x_M$. Note that x_M is a symplectic vector field $\Leftrightarrow$ the contraction $\iota_{x_M}\omega$ is a closed 1-form. If it's exact, then $\exists\, H_x \in C^\infty(M)$ w. $\iota_{x_M}\omega = dH_x$. We want to be able to choose H_x's in a nice way:

Definition: By a ==Hamiltonian== K-action on M we mean an action as above together w. a K-equivariant linear map (==comoment map==) $x \mapsto H_x : \mathfrak{k} \to C^\infty(M)$ s.t.

$$(3):\quad \iota_{x_M}\omega = dH_x \quad \forall x \in \mathfrak{k}$$

We define the moment map $\mu : M \to \mathfrak{k}^*$ by

$$\langle \mu(m), x\rangle := H_x(m).$$

It is K-equivariant.

Example: Let V be a vector space and $\omega \in \Lambda^2 V^*$, non-degenerate. Then we claim that we can take $H_x(v) = \frac{1}{2}\omega(xv, v)$. Indeed, the map is K-equivariant: $[kH_x](v) = \frac{1}{2}\omega(xk^{-1}v, k^{-1}v) = \frac{1}{2}\omega(\mathrm{Ad}(k)xv, v) = H_{\mathrm{Ad}(k)x}(v)$ & $[d_v H_x](u) = \frac{1}{2}(\omega(xv,u) + \omega(v, xu)) = [x \text{ is symplectic}]$

$= \omega(xv,u) = [\iota_{x_V}\omega](u)$, which checks (3). In particular, for $\omega = -2\cdot \mathrm{Im}(\cdot,\cdot)$ we recover $\mu(v): x \mapsto \sqrt{-1}\,(xv,v)$

In our proof of Theorem we will need the following general property of a moment map.

Lemma: Let μ be a moment map for a Hamiltonian action of K on (M,ω). Then $\forall\, x,y\in\mathfrak{k}$ & $m\in M$ we have

$$\omega_m(x_{M,m}, y_{M,m}) = \langle \mu(m), [x,y]\rangle.$$

Proof:

$$\omega_m(x_{M,m}, y_{M,m}) = [\iota_{x_M}\omega]_m(y_{M,m}) = [(3)] = d_m H_x(y_{M,m}) =$$
$\langle d_m\mu(y_{M,m}), x\rangle =$ [μ is K-equivariant, hence $\mathfrak{k}$-equivariant: intertwines y_M w. $y_{\mathfrak{k}^*}$] $= \langle \underbrace{y_{\mathfrak{k}^*,\mu(m)}}_{\in\mathfrak{k}^*}, x\rangle =$ [def'n of coadjoint action] $= -\langle \mu(m), [y,x]\rangle$. □

2.2) Proof of Theorem.

Since G/H is affine, it embeds as a closed G-stable subvariety into some G-rep'n V. Let v be the image of H. Take $(\cdot,\cdot)$, μ as in Sec. 1. We can replace v w. a G-conjugate & achieve $\mu(v)=0$, thx to Kempf-Ness thm.

Since $\mathfrak{g} = \mathbb{C}\otimes_{\mathbb{R}}\mathfrak{k}$, we have $\mathbb{C}\otimes_{\mathbb{R}}(\mathfrak{k}\cap\mathfrak{h}) \hookrightarrow \mathfrak{h} \Rightarrow$

(4) $\dim_{\mathbb{R}}(\mathfrak{k}\cap\mathfrak{h}) \leq \dim_{\mathbb{C}}\mathfrak{h}$.

Note that H is reductive $\Leftrightarrow H^\circ$ is. To prove that H° is reductive, it suffices to show that $H^\circ \cap K$ is Zariski dense in H°, which would follow if we show (4) is an equality, cf. Lemma in Sec. 1.3 of Lec 2. So it remains to show

(5) $\dim_{\mathbb{R}}(\mathfrak{k}\cap\mathfrak{h}) \geqslant \dim_{\mathbb{C}}\mathfrak{h} \Leftrightarrow \dim_{\mathbb{R}}\mathfrak{k}.v \leqslant \frac{1}{2}\dim_{\mathbb{R}}\mathfrak{g}.v$

Note that:

(i) $\mathfrak{g}.v$ is a complex subspace, hence $\omega = -2\,\mathrm{Im}(\cdot,\cdot)$ is non-degenerate on $\mathfrak{g}.v$.

(ii) By Lemma in Sec. 2.1, $\omega_v(x.v, y.v) = \langle \mu(v), [x,y]\rangle = 0\ \forall x,y\in\mathfrak{k} \Rightarrow \mathfrak{k}.v$ is an isotropic subspace of $\mathfrak{g}.v$. This yields the equivalent formulation of (5) & finishes the proof. $\square$

2.3) Luna's closedness criterion.

Thm (Luna): Let G be a reductive group acting on an affine variety X. Let $H\subset G$ be reductive & $x\in X^H$. If $N_G(H)x$ is closed, then so is Gx.

Proof: $\exists$ G-equivariant closed embedding of X into a rational representation V so we can assume $X=V$. We can choose maximal compact subgroups $K_H\subset K_N\subset K$ in $H\subset N:=N_G(H)\subset G$. Choose a K-invariant Hermitian scalar product $(\cdot,\cdot)$ on V & form the moment maps μ_N, μ for K_N, K. They are related as follows: if

$p: \mathfrak{k}^* \twoheadrightarrow \mathfrak{k}_N^*$ is the restriction map, then $\mu_N = p\circ\mu$. Note that x is fixed by K_H, hence $\mu(x)\in(\mathfrak{k}^*)^{K_H}$. Also

$Z_K(K_H)\subset K_N \Rightarrow \mathfrak{k}^{K_H}\subset\mathfrak{k}_N \Rightarrow [(\mathfrak{k}/\mathfrak{k}_N)^*]^{K_H}=0 \Rightarrow p|_{(\mathfrak{k}^*)^{K_H}}$ is injective.

Since $\mu(x)\in(\mathfrak{k}^*)^{K_H}$, we have $\mu_N(x)=0 \Rightarrow \mu(x)=0$.

Now we are done by the Kempf-Ness theorem $\square$

Corollary: If $N_G(H)/H$ is finite (e.g. H is a maximal torus), then Gx is closed.

Remark: In fact, the opposite inclusion in Thm is true as well: if Gx is closed, then $N_G(H)x$ is closed. This can be deduced from Vinberg's lemma (Sec 3.1 of Lec 6) and is left as an exercise.

2.4) Characteristic of a nilpotent element in $\mathfrak{g}$.

Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$ & $(\cdot,\cdot)_{ad}$ denote the Killing form. Let $e\in\mathfrak{g}$ be a nonzero nilpotent element. As was mentioned in Sec. 2.3.1 of Lec 10, we can include e into an $\mathfrak{sl}_2$-triple (e,h,f).

Proposition: h is a characteristic of e (w.r.t. $(\cdot,\cdot)_{ad}$) in the sense of Sec 2 of Lec 12.

Proof: Recall the Levi subgroup $G_0(h)=Z_G(h)$ & its normal subgroup $\underline{G}_0(h)$: the connected subgroup w. Lie algebra $\underline{\mathfrak{g}}_0(h)=\{x\in\mathfrak{g}_0(h) \mid (h,x)_{ad}=0\}$. As we've mentioned in Example in Sec. 1.1 of Lec 13, it's enough to show that $\underline{G}_0(h)e$ is closed.

Let S denote the connected subgroup of G w. Lie algebra $\mathrm{Span}_{\mathbb{C}}(e,h,f)\simeq \mathfrak{sl}_2$ (so that $S\simeq SL_2$ or SO_3). Let K_S denote the image of SU_2 under $SL_2 \twoheadrightarrow S$, it's a maximal compact subgroup. One can find K w. $K_S\subset K$.

Let τ be the anti-linear automorphism of $\mathfrak{g}$ that is the identity on $\mathfrak{k}$. It's restriction to $\mathfrak{s}=\mathrm{Span}_{\mathbb{C}}(e,h,f)\simeq\mathfrak{sl}_2$ fixes $\mathfrak{su}_2$, hence is given by $x\mapsto -\bar{x}^t$. Therefore

(*) $\quad \tau(e)=-f,\ \tau(f)=-e,\ \tau(h)=-h.$

On the other hand, $(\cdot,\cdot)_{ad}$ is negative definite on $\mathfrak{k}$ (b/c $\mathfrak{k}$ acts by skew-Hermitian operators on any rational representation of G). Hence $(\cdot,\cdot)_H := -(x,\tau(y))_{ad}$ is a Hermitian scalar product on $\mathfrak{g}$ invariant w.r.t. K. Observe that $\forall x\in\underline{\mathfrak{g}}_0(h)$:

(**) $\quad ([x,e],e)_H=[(*)]=([x,e],f)_{ad}=[\text{invariance}]=(x,h)=0$

Note that $K_e(h)=\mathrm{Stab}_K(\sqrt{-1}h)$ is a maximal compact subgroup of $G_0(h)$ (the latter is connected & $\mathrm{Lie}(K_0(h))$ is a real form of $\mathfrak{g}_0(h)$). Also $\underline{\mathfrak{k}}_0(h)=\mathfrak{k}_0(h)\cap\underline{\mathfrak{g}}_0(h)$ is a real form of $\underline{\mathfrak{g}}_0(h)$ so the Lie algebra of a maximal compact subgroup $\underline{K}_0(h)$ of

$\underline{G}_0(\mathfrak{h})$. Let μ denote the moment map from $\underline{K}_0(\mathfrak{h}) \curvearrowright \mathfrak{g}$. (**) says precisely that $\mu(e)=0$. So $\underline{G}_0(\mathfrak{h})e$ is closed in $\mathfrak{g}$ (& hence in $\mathfrak{g}_2(\mathfrak{h})$) by the Kempf-Ness theorem $\square$