Lecture 16, 3/5.

1) Quotients & slice theorem in symplectic setup.

Ref: [CdS], Part IX; [LS].

1.0) Reminder

We are in the setting of Sec 1.1 of Lec 15: G is a reductive group/$\mathbb{C}$ acting linearly on a vector space, $K\subset G$ is a maximal compact subgroup, and $(\cdot,\cdot)$ is a K-invariant Hermitian scalar product on V. We've introduced the map $\mu: V\to \mathfrak{k}^*$: $\langle \mu(v), x\rangle = \sqrt{-1}(xv,v)$. In Sec 2 we've seen that this a moment map for the Hamiltonian action of K on V (w. symplectic form $\omega(\cdot,\cdot) = -2\,\mathrm{Im}(\cdot,\cdot)$).

We proved the following:

Theorem (Kempf-Ness) Let $v\in V$.
1) $Gv\cap \mu^{-1}(0)\neq\varnothing \Leftrightarrow Gv$ is closed.
2) If $Gv\cap\mu^{-1}(0)\neq \varnothing$, then this intersection is a single K-orbit.

This theorem, in particular, serves as a motivation for defining a suitable version of quotients for Hamiltonian actions.

1.1) More on moment maps.

Let (M,ω) be a symplectic manifold, K be a Lie group acting on M with moment map $\mu: M \to \mathfrak{k}^*$. Recall that μ is a K-equivariant map s.t.

(1) $$\langle d_m\mu(u), x\rangle = \omega_m(x_{M,m}, u) \ \forall m\in M, u\in T_mM, x\in\mathfrak{k}$$

same as $d_m H_x(u)$, where $H_x = \langle\mu, x\rangle$

Here are some properties.

skew-orthogonal complement

Lemma: 1) $\ker d_m\mu = (T_mKm)^{\angle} := \{u\in T_mM \mid \omega(x_{M,m},u)=0\}$

2) $\operatorname{im} d_m\mu = (\mathfrak{k}/\operatorname{Lie}(\operatorname{Stab}_K(m)))^*$.

Proof: exercise: 1) & $\subset$ in 2) follow from (1) & $\supset$ in 2) follows from 1) & dimension count $\square$

Corollary: Suppose $m\in M$ is s.t. $\dim Km = \dim K$. Then m is a regular point of μ.

1.2) Hamiltonian reduction.

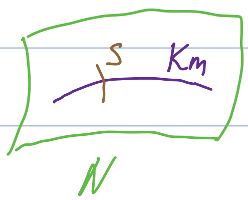
In the setting of the Kempf-Ness theorem, we see that the <u>sets</u> $\mu^{-1}(0)/K$ & $V/\!/G$ are in bijection. This (and other considerations) motivates us to consider a more general setting: let K

be a compact Lie group with Hamiltonian action on a symplectic manifold (M,ω). We want to understand what structure $\mu^{-1}(0)/K$ has. In this section we will consider a vanilla case when the K-action on $\mu^{-1}(0)$ is free.

Fact: Suppose K acts freely on a manifold N. Then the quotient set N/K has a unique C^∞-manifold structure making the map $N \twoheadrightarrow N/K$ a principal K-bundle (in the C^∞ category)

Sketch of proof:

Let S be a small transverse slice to a K-orbit in N:



Then the action map $K \times S \to N$ is an open embedding and S serves as a coordinate neighborhood in N/K. $\square$

Theorem (Marsden-Weinstein-Meyer)

Suppose we have a Hamiltonian action of K on (M,ω) s.t. the action of K on $\mu^{-1}(0)$ is free. Let $\iota: \mu^{-1}(0) \hookrightarrow M$ be the inclusion & $\pi: \mu^{-1}(0) \twoheadrightarrow \mu^{-1}(0)/K$ be the projection. Then $\exists!$ 2-form $\underline{\omega}$ on $\mu^{-1}(0)/K$ s.t. $\pi^*\underline{\omega} = \iota^*\omega$. This form is symplectic.

Proof: exercise. Hint: use Lemma & corollary from Sec 1.1

Remarks: 1) This theorem predates the Kempf-Ness theorem. The motivation comes from Classical Mechanics: the theorem implements the principle that the presence of continuous symmetries of a Hamiltonian system allows to reduce the dimension.

2) The question of what structure $\mu^{-1}(0)/K$ has when the freeness assumption is dropped is more complicated, see [LS] for some version of the answer. Note that this set has a natural topology: the finest topology s.t. $\mu^{-1}(0) \twoheadrightarrow \mu^{-1}(0)/K$ is continuous. One can show that every point has a neighborhood isomorphic to a neighborhood of a point in a variety, this will follow from a symplectic slice theorem to be covered in Sec 1.4 and other things, see Sec 5 in [LS].

1.3) Yet more on moment maps

Basic properties (of moment maps) - proofs are exercises.

1) If $\mu: M \to \mathfrak{k}^*$ is a moment map and $\chi \in (\mathfrak{k}^*)^K$, then $\mu+\chi$ (sending m to $\mu(m)+\chi$) is a moment map. Conversely, if M is connected & μ_1, μ_2 are moment maps for $K \curvearrowright M$, then $\mu_2 = \mu_1 + \chi$ for $\chi \in (\mathfrak{k}^*)^K$.

2) If (M_1,ω_1), (M_2,ω_2) are symplectic manifolds equipped w. Hamiltonian K-actions w. moment maps $\mu_i: M_i \to \mathfrak{k}^*$. Consider the product $M := M_1 \times M_2$ w. form $\omega = pr_1^*\omega_1 + pr_2^*\omega_2$ & diagonal action of K. Then $\mu: M \to \mathfrak{k}^*$, $\mu(m_1,m_2) = \mu_1(m_1)+\mu_2(m_2)$, is a moment map for $K \curvearrowright M$.

3) Let (M,ω) be a symplectic manifold w. a Hamiltonian action of K. Suppose that we are given a Lie group homomorphism $\varphi: L \to K$ & let φ denote the corresponding Lie algebra homomorphism. If $\mu: M \to \mathfrak{k}^*$ is a moment map for $K \curvearrowright M$, then $\varphi^* \circ \mu: M \to \mathfrak{l}^*$ is a moment map for $L \curvearrowright M$.

Example (cotangent bundle) Let M_0 be a manifold & $M = T^*M_0$ be its cotangent bundle (w. $\omega = d\beta$, where β is the canonical 1-form on M: if $\pi: M \twoheadrightarrow M_0$ is the projection, then $\beta_{(m,\alpha)}(\xi) = \langle \alpha, d_{(m,\alpha)}\pi(\xi)\rangle$ $m \in M_0$, $\alpha \in T^*_m M_0$, $\xi \in T_{(m,\alpha)}(M)$).

Note that we can view $Vect(M_0)$ as the space of fiberwise linear functions in $C^\infty(M)$. Let K be a Lie group acting on M_0. This induces the K-action on M by symplectomorphisms. It's Hamiltonian w. $H_x = x_{M_0} \in Vect(M_0) \subset C^\infty(M)$ (exercise).

Exercise: Let L be a Lie group, $K=L\times L$, $M_0=L$ w. actions of two copies of L from the left & from the right. We can identify $M=T^*L$ w. $L\times \mathfrak{l}^*$ using the trivialization by right-invariant vector fields. Show that $\mu: T^*L \to \mathfrak{l}^*\times\mathfrak{l}^*$ is given by $(\ell,\alpha)\mapsto(\ell.\alpha,-\alpha)$ $(\ell\in L, \alpha\in\mathfrak{l}^*)$.

1.4) Symplectic slice theorem

Let (M,ω) be a symplectic manifold equipped with a Hamiltonian action of K with moment map $\mu: M\longrightarrow \mathfrak{k}^*$. Given a point $m\in M$ we want to describe a neighborhood of Km & M as a Hamiltonian K-manifold. We first give a description of a K-manifold structure, the ordinary slice theorem, a much easier version of the result in Sec 1.5 of Lec 14.

Set $H:=\mathrm{Stab}_K(m)$. Note that for a manifold N w. H-action we can form the homogeneous bundle $K\times^H N=(K\times N)/H$ analogously to Sec 1.1 of Lec 14 but using Fact from Sec 1.2.

Now take N to be the normal space T_mM/T_mKm. Then $K\times^H N$ is nothing else but the normal bundle to Km in M. We have the following equivariant version of the tubular neighborhood theorem.

Lemma: There is a K-equivariant neighborhood of Km in M isomorphic to $K\times^H D$ for a small K-stable open disc $D\subset N$.

Our next task is to describe $T_m M$ up to H-equivariant linear symplectomorphism, this will inform our choice of a local model. For simplicity, we restrict to the case of $\mu(m)=0$. One can either do the general case similarly or reduce to this case, this will be addressed in a separate note.

Recall, Lemma in Sec 2.1 of Lec 15,

$$\omega_m(x_{M,m}, y_{M,m}) = \langle \mu(m), [x,y]\rangle = 0.$$

We write $\mathfrak{k}/\mathfrak{h}$ for $T_m Gm$. The pairing between $T_m Gm$ & $T_m M/(T_m Gm)^{\perp}$ is non-degenerate yielding an H-equivariant identification

$$T_m M/(T_m Gm)^{\perp} \xrightarrow{\sim} (\mathfrak{k}/\mathfrak{h})^*$$

Choose an H-equivariant section $(\mathfrak{k}/\mathfrak{h})^* \hookrightarrow T_m M$, so that the restriction of ω_m to $\mathfrak{k}/\mathfrak{h}\oplus(\mathfrak{k}/\mathfrak{h})^*$ is the natural form on this space. Let $V:=(\mathfrak{k}/\mathfrak{h}\oplus(\mathfrak{k}/\mathfrak{h})^*)^{\perp}$ & write ω_V for $\omega_m|_V$, this is an H-invariant symplectic form. So

$$T_m M = (\mathfrak{k}/\mathfrak{h})\oplus(\mathfrak{k}/\mathfrak{h})^*\oplus V \text{ \&}$$

(2) $\omega_m(y_1+z_1+v_1, y_2+z_2+v_2) = \langle y_2, z_1\rangle - \langle y_1, z_2\rangle + \omega_V(v_1, v_2).$

Our last step will be to produce a "model" Hamiltonian manifold. Namely consider the diagonal action of H on $T^*G\times V$, where the action on T^*G is from the right. It's Hamiltonian, see Sec 1.3, w. moment map described as follows. Set $\mu_V: V \to \mathfrak{h}^*, v \mapsto [x \mapsto \frac{1}{2}\omega_V(xv,v)]$. Then

$$\mu_H: T^*K \times V \longrightarrow \mathfrak{h}^*, (k,\alpha,v) \mapsto -\alpha|_{\mathfrak{h}} + \mu_V$$

We have a commuting K-action from the left; it has moment map $\mu_K: T^*K \times V \longrightarrow \mathfrak{k}^*, (k,\alpha,v) \mapsto k.\alpha$.

Consider the Hamiltonian reduction $M' = \mu_H^{-1}(0)/H$. The action of K on M' is Hamiltonian w. moment map induced from μ_K:

$$H.(k,\alpha,v) \mapsto k.\alpha.$$

The reason to consider M' is as follows. Let $m' = H.(1,0,0)$. Note that we have the projection $M' \longrightarrow G/H$, $H.(g,\alpha,v) \mapsto gH$. The fiber of $1H$ is $\{(\alpha,v) \in \mathfrak{g} \times V \mid \alpha|_{\mathfrak{h}} = \mu(v)\} \xrightarrow{\sim} (\mathfrak{g}/\mathfrak{h})^* \times V$ via $(\alpha,v) \mapsto (\alpha - \mu(v), v)$. Thanks to this we have an identification:

$$T_{m'}M' \xrightarrow{\sim} \underbrace{\mathfrak{g}/\mathfrak{h}}_{T_{m'}(Gm')} \oplus \underbrace{(\mathfrak{g}/\mathfrak{h})^* \oplus V}_{T_{m'}(\text{fiber})}$$

Exercise: Show that $\omega_{m'}$ is given by (2).

The following is the main result of this section.

Thm: $\exists$ K-stable open neighborhoods M_0 of m in M & M_0' of m' in M' s.t. $\exists$ K-equivariant symplectomorphism between M_0 & M_0' intertwining the moment maps.

Idea of proof:

1) Establish a suitable K-equivariant diffeomorphism $M_0 \xrightarrow{\varphi} M_0'$ using Lemma. One achieves $\omega_m = \varphi^*(\omega_m')$ w. suitable choice of φ).

2) Use Moser's trick: we have $\omega' = \varphi^*\omega$ at pts of $K.m$. From here using the method explained in Sec 7 of [CdS] one canonically produces a flow γ_t, $t \in [0,1]$, from $\omega', \varphi^*\omega$ defined on some neighborhood of Km s.t. $\gamma_1^*\omega' = \varphi^*\omega$.

Since both $\omega', \varphi^*\omega$ are K-invariant, the flow is K-equivariant & the neighborhood, where it's defined is K-invariant. So we get a K-equivariant symplectomorphism as in the Lemma. To show it intertwines μ & μ' note that $\mu - \mu'$ is constant see Sec 1.3 & note that $\mu_m = \mu_m' = 0$ $\square$

Corollary: A neighborhood of Km in $\mu^{-1}(0)/K$ is homeomorphic to a neighborhood of 0 in $\mu_V^{-1}(0)/H$.

Sketch of proof: Thx to Thm, we can replace M w. M'. Write $///_0$ for Hamiltonian reduction at 0, e.g. $M///_0 K := \mu^{-1}(0)/K$. The details of the following manipulation are left as an exercise:

$$[(T^*K \times V)///_0 H]///_0 \xrightarrow{\sim} [T^*K \times V]///_0 (G \times H) \xrightarrow{\sim} [(T^*K \times V)///_0 K]///_0 H$$
$$\xrightarrow{\sim} [(T^*K)///_0 K \times V]///_0 H = V///_0 H. \qquad \square$$

Exercise (in Linear algebra) $\exists$ H-invariant Hermitian scalar product $(\cdot,\cdot)$ on V s.t. $\omega_V = -2\,\mathrm{Im}(\cdot,\cdot)$.

In particular, $\mu_V^{-1}(0)/H \xrightarrow{\sim} V/\!/H_{\mathbb{C}}$, where $H_{\mathbb{C}}$ is the complexification.