Lecture 17, 03/24/25

1) GIT quotients

Refs: [PV], Sec 4.6; [MF], § 1.4.

1) GIT quotients

1.0) Introduction

Our base field is $\mathbb{C}$ (more generally, we can work w. any algebraically closed characteristic 0 field).

In this lecture we start studying GIT quotients. Here are two important features:

1) A GIT quotient is defined for an action of a reductive group G on an affine variety (or, more generally, affine finite type scheme) X together w. an auxiliary datum: a character $\theta: G \to \mathbb{C}^\times$. For trivial θ we recover the categorical quotient $X/\!\!/G$ from Lec 3. The notation in the general case is $X/\!\!/^\theta G$. The scheme $X/\!\!/^\theta G$ will naturally be a projective scheme over $X/\!\!/G$.

2) We have a G-stable open subset of "θ-semistable points" $X^{\theta\text{-}ss} \subset X$ & a G-invariant morphism $\pi^\theta: X^{\theta\text{-}ss} \longrightarrow X/\!\!/^\theta G$ (so that for trivial θ we have $X^{\theta\text{-}ss} = X$, $\pi^\theta = \pi$) that is surjective, affine & s.t. every fiber of π^θ contains a unique G-orbit that is closed in $X^{\theta\text{-}ss}$. In this way, $X/\!\!/^\theta G$ parameterizes the G-orbits that are

closed in $X^{\theta-ss}$.

1.1) Reminder on relative Proj.

The construction of $X/\!\!/^{\theta}G$ is based on the relative Proj construction (and can be viewed as the noncommutative generalization of the latter). So we first recall the Proj construction.

Let A be a $\mathbb{Z}_{\geqslant 0}$-graded finitely generated commutative algebra, $A=\bigoplus_{i\geqslant 0} A_i$. We write Y for $\operatorname{Spec}(A)$. For a homogeneous element $f\in A_i$ w. $i>0$ consider the principal open subscheme Y_f

Note that $\mathbb{C}[Y_f]=A[f^{-1}]$ is naturally $\mathbb{Z}$-graded (w. $\mathbb{C}[Y_f]_j=\{\frac{h}{f^k}\mid h\in A_{j+ik}\}$). The torus $\mathbb{C}^\times$ acts on A by automorphisms (so that $t.h=t^j h$ for $h\in A_j$). This gives rise to a $\mathbb{C}^\times$-action on Y & Y_f is $\mathbb{C}^\times$-stable.

Form $Y_f/\!\!/\mathbb{C}^\times=\operatorname{Spec}(\mathbb{C}[Y_f]_0)$. Our goal is to glue these affine schemes together.

Observation: Let $f\in A_i, g\in A_j$. Then $Y_{fg}/\!\!/\mathbb{C}^\times$ is identified w. the principal open subset in $Y_f/\!\!/\mathbb{C}^\times$ defined by $g^i/f^j\in\mathbb{C}[Y_f]_0$. This is b/c the algebras $A[(fg)^{-1}]_0$ & $A[f^{-1}]_0[(g^i/f^j)^{-1}]$ are equal as subalgebras of $A[(fg)^{-1}]$

So we can glue the affine schemes $Y_f /\!/ \mathbb{C}^\times$ together along the above identifications. In general such gluing (a special case of a colimit) fails to give a (separated) scheme but in our case it does. Namely, there is $e \in \mathbb{Z}_{>0}$ s.t. $A_{(e)} := \bigoplus_{i \geq 0} A_{ie}$ is generated by A_e as an A_0-algebra. Pick generators $F_1, \dots F_k$ of the A_0-module A_e and consider the closed subscheme $\mathrm{Proj}(A) \subset Y \times \mathbb{P}^{k-1}$ defined by the following equations:

$P([y_1 : \dots : y_k]) = 0 \quad \forall P \in A_0[x_1, \dots x_k]$ w. $P(F_1, \dots F_k) = 0$ in A, where $y_1, \dots y_k$ are the homogeneous coordinates on $\mathbb{P}^{k-1}$.

This subscheme is obtained by gluing $Y_f /\!/ \mathbb{C}^\times$ as explained above. It comes w. a projective morphism $\mathrm{Proj}(A) \longrightarrow \mathrm{Spec}(A) = Y /\!/ \mathbb{C}^\times$.

Remark: The construction above identifies $\mathrm{Proj}(A)$ & $\mathrm{Proj}(A_{(e)})$ $\forall e \in \mathbb{Z}_{>0}$.

1.2) Construction of GIT quotient.

Let G be a reductive group & $\theta: G \to \mathbb{C}^\times$ be a character. Let $\mathbb{C}_\theta$ denote the 1-dimensional G-representation corresponding to θ: $g.v = \theta(g)v \ \forall g \in G, v \in \mathbb{C}_\theta$. Let z denote the coordinate on $\mathbb{C}_\theta$ so that $g.z = \theta(g)^{-1}z$. Consider the algebra $A := \mathbb{C}[X \times \mathbb{C}_\theta]^G$. Note that:

(i) A is finitely generated by Prop 1 in Sec 1.0 of Lec 3

(ii) A is graded (by degree in z). More precisely, set

$$\mathbb{C}[X]^{G,n\theta} = \{f \in \mathbb{C}[X],\ g.f = \theta(g)^n f\}$$

(elements of $\mathbb{C}[X]^{G,n\theta}$ are called $n\theta$-semiinvariants). Then

$$\mathbb{C}[X \times \mathbb{C}_\theta]^G = \bigoplus_{n \geq 0} \mathbb{C}[X]^{G,n\theta} z^n$$

Moreover the grading on A comes from $\mathbb{C}^\times \curvearrowright X \times \mathbb{C}_\theta$, $t.(x,v) = (x, t^{-1}v)$ commuting w. G.

Definition (Mumford): The GIT quotient $X/\!\!/^\theta G$ is $\operatorname{Proj}(A)$.

Note that $X/\!\!/G = \operatorname{Spec}(A_0)$ so we indeed get a projective morphism $X/\!\!/^\theta G \longrightarrow X/\!\!/G$.

Note also that, for $n > 0$, we have $X/\!\!/^{n\theta} G = X/\!\!/^\theta G$ thx to the remark in Sec 1.1 (here we write $n\theta$ for the character sending g to $\theta(g)^n$).

Example: We recover the construction in the previous section as follows. Let $G = \mathbb{C}^\times$ act on X so that the induced grading on $\mathbb{C}[X]$ is nonnegative. Identify the character lattice $\mathfrak{X}(\mathbb{C}^\times)$ w. $\mathbb{Z}$ via $n \in \mathbb{Z} \mapsto [t \mapsto t^n] \in \mathfrak{X}(\mathbb{C}^\times)$. If $\theta = 1$, then $\mathbb{C}[X \times \mathbb{C}_\theta]^{\mathbb{C}^\times} \xrightarrow{\sim} \mathbb{C}[X]$ as graded algebras (exercise). If $\theta = 0$, then $\mathbb{C}[X \times \mathbb{C}_\theta]^{\mathbb{C}^\times} = [\mathbb{C}^\times \curvearrowright \mathbb{C}_\theta$

is trivial$] = \mathbb{C}[X]^{\mathbb{C}^\times} \otimes \mathbb{C}[z]$, where the first factor is in degree 0.

So $X/\!/^{\theta} G \xrightarrow{\sim} \mathrm{Proj}\left(\mathbb{C}[X]^{\mathbb{C}^\times} \otimes \mathbb{C}[z]\right) = X/\!/\mathbb{C}^\times \times \underbrace{\mathrm{Proj}\left(\mathbb{C}[z]\right)}_{pt} = X/\!/\mathbb{C}^\times$

And if $\theta < 0$, then both $\mathbb{C}[X]$ & $\mathbb{C}[z]$ are $\mathbb{Z}_{\geq 0}$-graded, so $\left(\mathbb{C}[X] \otimes \mathbb{C}[z]\right)^{\mathbb{C}^\times} = \mathbb{C}[X]^{\mathbb{C}^\times} \otimes \mathbb{C}$ (in deg 0). Here $\mathrm{Proj}\left(\mathbb{C}[X]^{\mathbb{C}^\times}\right) = \emptyset$.

Exercise: For general G, show that $X/\!/^{id} G \xrightarrow{\sim} X/\!/G$.

1.3) $X^{\theta\text{-ss}}$ & π^{θ}

Let $X^{\theta\text{-ss}} = \{x \in X \mid \exists\, n > 0 \text{ \& } f \in \mathbb{C}[X]^{G, n\theta} \text{ s.t. } f(x) \neq 0\}$, equivalently $X^{\theta\text{-ss}} = \bigcup_f X_f$, where f runs over $\mathbb{C}[X]^{G,n\theta}$ w. $n > 0$. A point in $X^{\theta\text{-ss}}$ is called θ-semistable.

Exercise: Show that in the setting of the previous example:

- $X^{\theta\text{-ss}}$ is the complement of the zero set of the ideal $\bigoplus_{i>0} \mathbb{C}[X]_i \subset \mathbb{C}[X]$ if $\theta > 0$
- $X^{\theta\text{-ss}} = X$ if $\theta = 0$
- $X^{\theta\text{-ss}} = \emptyset$ if $\theta < 0$.

Note that for $f \in \mathbb{C}[X]^{G,n\theta}$, we have $f(gx) = \theta(g)^{-n} f(x)$ so X_f (and hence $X^{\theta\text{-ss}}$) is G-stable. Let $\pi : X \to X/\!/G$ & $\pi_f : X_f \to X_f/\!/G$ denote the quotient morphisms so that we have the following commutative diagram (see Sec 1.3 in Lec 3):

$$\begin{tikzcd} X_f \arrow[r, hook] \arrow[d, "\pi_f"'] & X \arrow[d, "\pi"] \\ X_f/\!\!/G \arrow[r] & X/\!\!/G \end{tikzcd} \qquad (1)$$

A relevance to the construction in Sec 1.1 is as follows. A homogeneous element of degree n in $A = \mathbb{C}[X\times\mathbb{C}_\theta]^G$ is of the form fz^n with $f\in\mathbb{C}[X]^{G,n\theta}$. We have $A[(fz^n)^{-1}] = (\mathbb{C}[X\times\mathbb{C}_\theta][(fz^n)^{-1}])^G$
$= (\mathbb{C}[X\times\mathbb{C}_\theta][f^{-1}, z^{-1}])^G = \mathbb{C}[X_f\times\mathbb{C}_\theta^\times]^G$. Now $A[(fz^n)^{-1}]_0 =$
$= \mathbb{C}[X_f\times\mathbb{C}_\theta^\times]^{G\times\mathbb{C}^\times} =$ [$\mathbb{C}^\times$ acts trivially on X_f & transitively on $\mathbb{C}_\theta^\times$]
$= \mathbb{C}[X_f]$. So $X_f = \operatorname{Spec}(A[(fz^n)^{-1}]_0)$ embeds into $X/\!\!/^\theta G$
$= \operatorname{Proj}(A)$ as an open subscheme. Moreover, the intersection of two subschemes, $X_f/\!\!/G$ & $X_h/\!\!/G$, is $X_{fh}/\!\!/G$ by Observation in Sec 1.1.

Proposition: $\exists!$ morphism $\pi^\theta: X^{\theta\text{-}ss}\to X/\!\!/^\theta G$ making the following diagram commutative $\forall\, n>0$ & $f\in\mathbb{C}[X]^{G,n\theta}$

$$\begin{tikzcd} X_f \arrow[r, hook] \arrow[d, "\pi_f"'] & X^{\theta\text{-}ss} \arrow[r, hook] \arrow[d, "\pi^\theta"] & X \arrow[d, "\pi"] \\ X_f/\!\!/G \arrow[r, hook] & X/\!\!/^\theta G \arrow[r] & X/\!\!/G \end{tikzcd}$$

Moreover,

(a) π^θ is surjective

(b) $(\pi^\theta)^{-1}(X_f/\!\!/G) = X_f \Leftrightarrow$ the left square is Cartesian.

(c) every fiber of π^θ contains a unique G-orbit that is closed in $X^{\theta\text{-}ss}$.

Proof:

The uniqueness of π^θ will follow since the open subsets X_f cover $X^{\theta\text{-ss}}$ & $\pi^\theta|_{X_f}$ must be π_f. To show that π^θ exists it's enough to show that $\pi_f|_{X_f\cap X_h} = \pi_h|_{X_f \cap X_h}$ $\forall f\in \mathbb{C}[X]^{G,n\theta}$ & $h\in \mathbb{C}[X]^{G,m\theta}$ $(n,m>0)$. But $X_f\cap X_h = X_{fh}$. So we just need to show that the following is commutative

$$\begin{tikzcd} X_{fh} \arrow[r, hook] \arrow[d, "\pi_{fh}"'] & X_f \arrow[d, "\pi_f"] \\ X_{fh}/\!\!/G \arrow[r, hook] & X_f/\!\!/G \end{tikzcd}$$

but this follows from the general result in Sec 1.3 in Lec 3.

(a): The morphism π^θ is surjective b/c each π_f is surjective & the open subsets $X_f/\!\!/G$ cover $X/\!\!/^\theta G$ (by the general results in Sec 1.1).

(b): Suppose $\pi^\theta(x)\in X_f/\!\!/G$ but $x\notin X_f \Leftrightarrow f(x)=0$. Pick $h\in\mathbb{C}[X]^{G,m\theta}$ w. $x\in X_h$. Then $\pi^\theta(x)\in X_f/\!\!/G \cap X_h/\!\!/G = X_{fh}/\!\!/G$. But $\pi^\theta(x) = \pi_h(x)$ & $\frac{f^m}{h^n}\in \mathbb{C}[X_h]^G$ vanishes on x. This shows $\pi_h(x)\notin X_{fh}/\!\!/G$, a contradiction.

(c): Let $\mathcal{X}\in X/\!\!/^\theta G$. Note that $(\pi^\theta)^{-1}(\mathcal{X})$ contains an orbit of minimal dimension. Such an orbit is closed in $(\pi^\theta)^{-1}(\mathcal{X})$, hence in $X^{\theta\text{-ss}}$. On the other hand, let $f\in\mathbb{C}[X]^{G,n\theta}$ be s.t. $\mathcal{X}\in X_f/\!\!/G$. By (b), $(\pi^\theta)^{-1}(\mathcal{X})\subset X_f/\!\!/G$, so $(\pi^\theta)^{-1}(\mathcal{X}) = (\pi_f)^{-1}(\mathcal{X})$. From Lec 3 we know that $(\pi_f)^{-1}(\mathcal{X})$ contains a unique closed G-orbit $\square$

Remark: (b) shows that π^θ is an affine morphism.

The following lemma gives an equivalent condition for semi-stability. We will use it in the next lecture for a Hilbert-Mumford type statement.

Lemma: a) For $x \in X$ TFAE:

(1) $x \in X^{\theta-ss}$

(2) $\overline{G(x,1)} \cap (X \times \{0\}) = \emptyset$ in $X \times \mathbb{C}_\theta$.

Proof:

(1) $\Rightarrow$ (2): $x \in X^{\theta-ss} \Leftrightarrow \exists\, f \in \mathbb{C}[X]^{G, n\theta}$ w. $n>0$ & $f(x)=1$. Let $F = fz^n \in \mathbb{C}[X \times \mathbb{C}_\theta]^G$. Then $F(x,1) \neq 0$. On the other hand, $F|_{X \times \{0\}} = 0$ & (2) follows.

(2) $\Rightarrow$ (1): Let $\tilde{\pi}: X \times \mathbb{C}_\theta \twoheadrightarrow (X \times \mathbb{C}_\theta)/\!/G$ denote the quotient morphism. By Sec 1.4 of Lec 3, if $Y_1, Y_2 \subset X \times \mathbb{C}^\theta$ are closed G-stable subvarieties w. $Y_1 \cap Y_2 = \emptyset$, then $\tilde{\pi}(Y_1) \cap \tilde{\pi}(Y_2) = \emptyset$. Apply this to $Y_1 = \overline{G.(x,1)}$ & $Y_2 = X \times \{0\}$. We get that $\tilde{\pi}(x,1) \notin \tilde{\pi}(X \times \{0\})$. So we can find $F \in \mathbb{C}[X \times \mathbb{C}_\theta]^G$ w. $F|_{X \times \{0\}} = 0$ & $F(x,1) \neq 0$. We can write F as $\sum f_n z^n$ w. $f_n \in \mathbb{C}[X]^{G, n\theta}$. Since $F|_{X \times \{0\}} = 0$, we see that $f_0 = 0$. From $F(x,1) \neq 0$ we deduce $\exists\, n>0$ s.t. $f_n(x) \neq 0 \Rightarrow x \in X^{\theta-ss}$ $\square$