Lecture 18, 03/26/25

1) Hilbert-Mumford type theorems
2) Examples

Ref: [MF], Sec 2.1.

1.0) Reminder.

Let G be a reductive group & $\theta: G \to \mathbb{C}^\times$ be a character. Let $\mathbb{C}_\theta$ denote the 1-dimensional G-representation corresponding to θ.

Let G act on a finite type affine scheme X. In Lec 17, we defined the GIT-quotient $X/\!\!/^\theta G$ as the Proj of the graded algebra $\mathbb{C}[X\times\mathbb{C}_\theta]^G \xrightarrow{\sim} \bigoplus_{n\geqslant 0} \mathbb{C}[X]^{G,n\theta}$; where $\mathbb{C}[X]^{G,n\theta}=\{f\in \mathbb{C}[X] \mid g.f=\theta(g)^n f\}$. We defined the locus of θ-semistable points

$$X^{\theta\text{-}ss}=\bigcup_f X_f, \text{ where } f \text{ runs over } \mathbb{C}[X]^{G,n\theta}\ \forall n>0.$$

We have constructed a G-invariant morphism $\pi^\theta: X^{\theta\text{-}ss}\to X/\!\!/^\theta G$ s.t. the following diagram is commutative

$$(*)\qquad \begin{tikzcd} X_f \arrow[r, hook] \arrow[d, "\pi_f"'] & X^{\theta\text{-}ss} \arrow[r, hook] \arrow[d, "\pi^\theta"] & X \arrow[d, "\pi"] \\ X_f/\!\!/G \arrow[r, hook] & X/\!\!/^\theta G \arrow[r] & X/\!\!/G \end{tikzcd}$$

Moreover, π^θ is surjective, the left square is Cartesian & every fiber of π^θ contains a unique closed G-orbit.

Further, in Lec 17 we have proved the following lemma.

Lemma: a) For $x\in X$ TFAE:

(1) $x\in X^{\theta-ss}$

(2) $\overline{G(x,1)}\cap(X\times\{0\})=\emptyset$ in $X\times\mathbb{C}_\theta$.

1.1) Hilbert-Mumford type theorems

We would like to understand a criterium for $x\in X$ to lie in $X^{\theta-ss}$ & for an orbit of x to be closed there. For this we will state and prove statements similar in spirit to the Hilbert-Mumford theorem from Lec 11.

Note that for $\theta: G\to\mathbb{C}^\times$ & $\gamma:\mathbb{C}^\times\to G$ we can consider their pairing $\langle\theta,\gamma\rangle\in\mathbb{Z}$ defined by $\theta\circ\gamma(t)=t^{\langle\theta,\gamma\rangle}$.

Theorem: 1) For $x\in X$ TFAE:

(a) $x\in X^{\theta-ss}$

(b) If $\lim\limits_{t\to 0}\gamma(t)x$ exists in X, then $\langle\theta,\gamma\rangle\leqslant 0$.

2) Suppose that $x\in X^{\theta-ss}$ & $y\in X^{\theta-ss}$ are s.t. $Gy\subset\overline{Gx}$ & Gy is closed in $X^{\theta-ss}$. Then $\exists$ 1-parameter subgroup $\gamma:\mathbb{C}^\times\to G$ s.t. $\langle\theta,\gamma\rangle=0$ & $\lim\limits_{t\to 0}\gamma(t)x$ exists and lies in Gy.

1) is often called the Hilbert-Mumford criterium for semi-stability.

Example: Suppose $G=\mathbb{C}^\times$ & $\mathbb{C}^\times \curvearrowright X$ is s.t. the resulting grading on $\mathbb{C}[X]$ is by $\mathbb{Z}_{\geq 0}$. The locus $X^{\theta-ss}$ was determined in Exercise from Sec 1.3 in Lec 17. Let's revisit this computation using the theorem.

Since the grading on $\mathbb{C}[X]$ is by $\mathbb{Z}_{\geq 0}$ we have (exercise):

(i) $\lim_{t\to 0} t.x$ exists $\Leftrightarrow$ $t.x=x$ $\forall t$ $\Leftrightarrow$ $f(x)=0$ $\forall f\in \bigoplus_{i>0}\mathbb{C}[X]_i$

(ii) $\lim_{t\to 0} t^{-1}.x$ exists always.

So, for $\theta>0$, $X^{\theta-ss}$ consists of all points not satisfying (i). For $\theta=0$, $X^{\theta-ss}$ consists of all points, while for $\theta<0$, $X^{\theta-ss}=\emptyset$. This recovers the conclusions of the exercise in Sec 1.3 of Lec 17.

Proof: 1): By Lemma in Sec 1.0, (a) $\Leftrightarrow$

(c) $\overline{G(x,1)}\cap X\times\{0\}=\emptyset$ in $X\times\mathbb{C}_\theta$ $\Leftrightarrow$

(c') the closed G-orbit in $\overline{G(x,1)}$ doesn't lie in $X\times\{0\}$.

Note that

(**) $$\lim_{t\to 0}\gamma(t)(x,1)=\lim_{t\to 0}(\gamma(t)x, t^{\langle\gamma,\theta\rangle})$$

exists iff $\lim_{t\to 0}\gamma(t)x$ exists & $\langle\gamma,\theta\rangle\geq 0$. Moreover, under these conditions, (**) lies in $X\times\{0\}$ iff $\langle\gamma,\theta\rangle>0$. To show that (b) $\Leftrightarrow$

(c) we combine (c') w. the Hilbert-Mumford theorem: $\exists\ \gamma$ s.t. $\lim_{t\to 0} \gamma(t).(x,1)$ exists in $X\times \mathbb{C}_\theta$ and lies in the unique closed G-orbit in the closure of $G.(x,1)$. Details are exercise.

2) Let $f\in \mathbb{C}[X]^{G,n\theta}$ w. $f(x)\neq 0$. Since $\pi^\theta(x)=\pi^\theta(y)$, we deduce $f(y)\neq 0$ from $\pi^\theta(x)\in X_f/\!/G$.

Exercise 1: Gy is the unique closed orbit in the closure of Gx in X_f. Hint: use that the left square of (*) is Cartesian.

Note that $f(\gamma(t)x)=[f(g^{-1}x)=\theta(g)^{-n}f(x)]=t^{-n<\gamma,\theta>}f(x)$ has nonzero limit iff $<\gamma,\theta>=0$.

So the following two conditions are equivalent:

(i) $\lim_{t\to 0}\gamma(t)x$ exists in X_f

(ii) $\lim_{t\to 0}\gamma(t)x$ exists in X & $f(\lim_{t\to 0}\gamma(t)x)\neq 0 \Leftrightarrow <\gamma,\theta>=0$.

Again, thx to this equivalence we deduce 2) from the Hilbert-Mumford theorem applied now to the action of G on X_f (exercise). $\square$

The following exercise will be useful in the next lecture.

Exercise 2: Use 2) of Thm (and its proof) to show that TFAE:

(a) For $x\in X^{\theta\text{-}ss}$, the orbit Gx is closed in $X^{\theta\text{-}ss}$

(b) $G.(x,1)$ is closed in $X\times\mathbb{C}_\theta$.

2) Examples

Example 1: Let V be a finite dimensional vector space, $k\in\mathbb{Z}_{>0}$, $X=\operatorname{Hom}_{\mathbb{C}}(\mathbb{C}^k,V)$, $G=GL_k$ acting on X by $g.x=xg^{-1}$. Let $\theta=\det$. We claim that

(a) $x\in X^{\theta\text{-ss}}\Leftrightarrow x$ is injective &

(b) $X/\!/^{\theta}G\xrightarrow{\sim}Gr(k,V)$.

First, we need to understand when $\lim\limits_{t\to 0}x\gamma(t)^{-1}$ exists for $\gamma:\mathbb{C}^\times\to GL_k$.

Exercise (also useful for the homework!)

Let $\mathbb{C}^k_i(\gamma)=\{u\in\mathbb{C}^k\mid\gamma(t)u=t^iu\}$. Set $\mathbb{C}^k_{>0}(\gamma):=\bigoplus\limits_{i>0}\mathbb{C}^k_i(\gamma)$. TFAE

- $\lim\limits_{t\to 0}x\gamma(t)^{-1}$ exists
- $\ker x\supset\mathbb{C}^k_{>0}(\gamma)$.

Note that $\mathbb{C}^k_{>0}(\gamma)=0\Rightarrow\langle\theta,\gamma\rangle\leqslant 0$. So, by Theorem, if x is injective, then it's θ-semistable. Conversely, if $\ker x\neq 0$ we can choose a complementary subspace $U\subset\mathbb{C}^k$ & define $\gamma(t)$ acting by t on $\ker x$ & trivially on U. Then $\lim\limits_{t\to 0}x\gamma(t)^{-1}$ exists but $\langle\theta,\gamma\rangle=\dim\ker x>0$. Theorem shows that x is not θ-semistable. This finishes the proof of (a).

Let's establish (6). We note that the map $X^{\theta\text{-}ss} \longrightarrow Gr(k,V)$, $x \mapsto \ker x$ is a morphism (exercise in Plücker charts) & each fiber is a $GL(k)$-orbit (exercise in Linear algebra). For each $f \in \mathbb{C}[X]^{G, n\theta}$ w. $n > 0$, $x \mapsto \ker x : X_f \longrightarrow Gr(k,V)$ descends to a morphism $X_f /\!/ G \longrightarrow Gr(k,V)$ by Problem 2 in HW1. The morphisms agree on $X_{fh} /\!/ G = X_f /\!/ G \cap X_h /\!/ G$ and so descend to $X /\!/^{\theta} G \longrightarrow Gr(k,V)$. We get a bijective morphism to a normal variety, hence an isomorphism. There are also at least two other ways to establish this isomorphism.

Remark: 1) It's easy to see that $\bigoplus_{n \geq 0} \mathbb{C}[X]^{G, n\theta}$ coincides with $\mathbb{C}[X]^{SL_k}$. The latter algebra can be computed: it equals to the homogeneous coordinate algebra of $Gr(k,V)$ for the Plücker embedding. This also shows that $X /\!/^{\theta} G = \mathrm{Proj}(\mathbb{C}[X]^{SL_k})$ but is more complicated: the description of $\mathbb{C}[X]^{SL_k}$ above requires knowing that the homogeneous coordinate ring is normal.

2) The description in 1) can be generalized as follows. Let X be a vector space & $G \subset GL(X)$ be a reductive subgroup containing the scaling torus. Let $G_0 := G \cap SL(X)$. Let θ be the restriction of $\det^{-1} : GL(X) \longrightarrow \mathbb{C}^{\times}$ to G (note the change of sign from the previous example). Then $X /\!/^{\theta} G = \mathrm{Proj}(\mathbb{C}[X]^{G_0})$. Moreover, $X^{\theta\text{-}ss}$ is

$X|\pi_{G_0}^{-1}(0)$ (exercise).

Example 2: We now consider a generalization of the previous example: representations of quivers.

By a quiver we mean an oriented graph $Q=(Q_0,Q_1,t,h)$, where Q_0 is a set of vertices, Q_1 is a set of arrows (= oriented edges) & $t,h: Q_1\to Q_0$ are tail & head maps: $\underset{t(a)}{\bullet}\xrightarrow{a}\underset{h(a)}{\bullet}$

A representation of Q is a collection of vector spaces V_i, $i\in Q_0$, and linear maps $x_a: V_{t(a)}\to V_{h(a)}$, $a\in Q_1$. We consider the case when all V_i are finite dimensional, so we can assign the dimension vector $v:=(\dim V_i)_{i\in Q_0}\in \mathbb{Z}_{\geq 0}^{Q_0}$. Let $Rep(Q,v)$ denote the set of representations of Q in fixed vector spaces V_i of dimension v so that $Rep(Q,v)=\bigoplus_{a\in Q_1} Hom(V_{t(a)},V_{h(a)})$. It is a vector space with an action of $GL(v):=\prod_{i\in Q_0} GL(V_i)$. Note that the 1-dimensional torus $\{(z\, Id_{V_i})\,|\,z\in\mathbb{C}^\times\}$ acts trivially on $Rep(Q,v)$ & so the action of $GL(v)$ factors through $PGL(v):=GL(v)/\{(z\, Id_{V_i})\}$.

For $v,\theta\in\mathbb{Z}^{Q_0}$ let $\theta\cdot v=\sum_{i\in Q_0}\theta_i v_i$. The character lattice of $PGL(v)$ is identified with $\{\theta\in\mathbb{Z}^{Q_0}\,|\,\theta\cdot v=0\}$ via

$$(\theta_i)_{i\in Q_0}\mapsto \Big[(g_i)\mapsto \prod_{i\in Q_0}\det(g_i)^{\theta_i}\Big]$$

For such $\theta\in\mathbb{Z}^{Q_0}$ we want to describe $Rep(Q,v)^{\theta\text{-}ss}$. Note that we can talk about subrepresentations of $(V_i)_{i\in Q_0}$: a collection $U_i \subset V_i$ w. $x_a U_{t(a)} \subset U_{h(a)}$ $\forall a \in Q_1$.

Proposition: $x \in Rep(Q,v)^{\theta-ss} \Leftrightarrow \forall$ subrepresentation (U_i) of (V_i) we have $\theta\cdot(\dim U_i)_{i\in Q_0} \leq 0$.

Proof: Again, we start by analyzing when $\lim_{t\to 0} \gamma(t)x$ exists. Set $V = \bigoplus_{i\in Q_0} V_i$. Choose a lift $\tilde{\gamma}$ of γ to $GL(v)$. For $n\in\mathbb{Z}$, let $V^n(\tilde{\gamma}) = \{u \in V \mid \tilde{\gamma}(t)u = t^n u\}$. The different lifts differ by a homomorphism to $\{(z\,Id_{V_i})\}$, so for a different choice of lift $\tilde{\gamma}'$, we have $V^n(\tilde{\gamma}') = V^{n+m}(\tilde{\gamma})$ w. $m\in\mathbb{Z}$. Hence we can assume that $V^n(\tilde{\gamma}) = \{0\}$ for $n<0$. For $n\geq 0$, set $V^{\geq n}(\tilde{\gamma}) = \bigoplus_{m=n}^{\infty} V^m(\tilde{\gamma})$. Similarly to Sec 1.4 of Lec 12 (of which the present setup is a special case), $\lim_{t\to 0}\gamma(t)x$ exists iff $V^{\geq n}(\tilde{\gamma}) = \bigoplus_{i\in Q_0} V_i^{\geq n}(\tilde{\gamma})$ is a subrepresentation for each $n\geq 0$. Let v^n denote the dimension vector of $V^n(\tilde{\gamma}')$ & $v^{\geq n} = \sum_{m=n}^{\infty} v^m$, dimension vector of $V^{\geq n}(\tilde{\gamma}')$. Then

$\langle\theta,\gamma\rangle$ = power of t in $\prod \deg(\tilde{\gamma}_i(t))^{\theta_i}$ = [$\tilde{\gamma}_i(t)$ has v_i^n eigenvalues t^n] $= \sum_{i\in Q_0}\sum_{n\geq 0} n v_i^n \theta_i = \sum_{n\geq 0} v^{\geq n}\cdot\theta$.

So if $u\cdot\theta \leq 0$ $\forall$ dimension vectors u of subrepresentations, then $\langle\theta,\gamma\rangle \leq 0$. Conversely, for any subrepresentation $(U_i)\subset(V_i)$ we can find $\tilde{\gamma}$ w. $V_i = V_i^{\geq 0}$, $U_i = V_i^{\geq 1}$, $\{0\} = V_i^{\geq 2}$. For such $\tilde{\gamma}$, we have $\langle\theta,\gamma\rangle = \theta\cdot(\dim U_i)$ finishing the proof. □

Remarks: 1) A connection to Example 1 is as follows: consider the quiver $\underset{1}{\circ} \overset{w}{\rightrightarrows} \underset{\infty}{\circ}$, where $w = \dim V$ & dimension vector $v = (k, 1)$. Then $GL(v) = GL(k) \times GL(1)$ & the inclusion $GL(k) \hookrightarrow GL(v)$ gives rise to an isomorphism $GL(k) \xrightarrow{\sim} PGL(v)$. Take θ of the form $(\alpha, -k\alpha)$ for $\alpha > 0$. Let $x = (x_a)_{a \in Q_1} : \mathbb{C}^k \longrightarrow \mathbb{C}^w$. There are two kinds kinds of subrepresentations:

- $(U_1, \mathbb{C})$ for $U_1 \subset \mathbb{C}^k$, they satisfy $(\dim U_i) \cdot \theta = (\dim U - k)\alpha \leq 0$
- $(U_1, \{0\})$ for $U_1 \subset \ker x$, they satisfy $(\dim U_i) \cdot \theta \leq 0 \Leftrightarrow U_1 = \{0\}$.

We recover the stability condition from Example 1.

2) More generally, for $\underset{1}{\circ} \to \underset{2}{\circ} \to \circ \to \dots \underset{\ell}{\circ} \overset{w}{\rightrightarrows} \underset{\infty}{\circ}$ dimension vector $v = (k_1, \dots, k_\ell, 1)$ & $\theta = (1, \dots, 1, -\sum k_i)$ we have $Rep(Q, v)^{\theta\text{-}ss} = \{$all maps $\mathbb{C}^{k_1} \to \mathbb{C}^{k_2} \to \dots \to \mathbb{C}^w$ are injective &

$$Rep(Q, v) /\!/^{\theta} GL(v) \xrightarrow{\sim} Fl(k_1, k_2, \dots k_\ell ; V)$$

In general, the GIT quotient of interest doesn't admit such an explicit description.