Lecture 19, 4/2/25.

1) GIT Hamiltonian reductions: $\mathrm{Hilb}_n(\mathbb{C}^2)$ & CM space.

Refs: [E], [Nak], [GG].

1) GIT Hamiltonian reductions: $\mathrm{Hilb}_n(\mathbb{C}^2)$ & CM space.

1.0) GIT Hamiltonian reduction.

We've considered Hamiltonian reductions for actions of compact groups on symplectic C^∞-manifolds. The same constructions work for actions of reductive groups on symplectic algebraic varieties (cf. Bonus to Lec 16). A basic example is a symplectic vector space V over $\mathbb{C}$ w. form $\omega_{\mathbb{C}} \in \Lambda^2 V^*$ & action of G coming from $G \to Sp(V)$. This action is Hamiltonian w. moment map $\mu: V \to \mathfrak{g}^*$ given by the same formula as in the compact case: $\langle \mu(v), x \rangle = \frac{1}{2}\omega(xv, v)$. So for $\lambda \in (\mathfrak{g}^*)^G$ & $\theta: G \to \mathbb{C}^\times$ we can consider the GIT Hamiltonian reduction $V///_\lambda^\theta G := \mu^{-1}(\lambda)//^\theta G$. If the G-action on $\mu^{-1}(\lambda)^{\theta-ss}$ is free then an algebraic version of the Marsden-Weinstein-Meyer thm from Sec 1.2 in Lec 16 shows that $V///_\lambda^\theta G$ carries an <u>algebraic</u> symplectic form: a unique 2-form $\underline{\omega}_{\mathbb{C}}$ s.t. $(\pi^\theta)^* \underline{\omega}_{\mathbb{C}} = \underline{\iota}^* \omega_{\mathbb{C}}$, where $\underline{\iota}: \mu^{-1}(0)^{\theta-ss} \hookrightarrow V$ is the inclusion. To rigorously construct $\underline{\omega}_{\mathbb{C}}$ is an extended exercise; hint: if G acts freely on $X^{\theta-ss}$, then π^θ is a principal G-bundle, this is b/c π^θ is glued from categorical quotients, for which the claim is Corollary in Sec 1.4 of Lec 14.

In this lecture we will consider special cases of this construction: Hilbert schemes of points on $\mathbb{C}^2$ & Calogero-Moser spaces.

1.1) Construction.

We are interested in V of a special form: let V_0 be a vector space. Set $V:=V_0\oplus V_0^*(=T^*V_0)$ & equip it with the following symplectic form:

$$\omega((v_1,\alpha_1),(v_2,\alpha_2))=\langle v_1,\alpha_2\rangle-\langle v_2,\alpha_1\rangle$$

- the natural form of T^*V_0. Let G act linearly on V_0. This action extends to the G-action on $V_0\oplus V_0^*$, which preserves ω.

Lemma 1: We have

$$\langle\mu(v,\alpha),x\rangle=\langle xv,\alpha\rangle \quad \forall v\in V_0, \alpha\in V_0^*, x\in\mathfrak{g}.$$

Proof:

$$\langle\mu(v,\alpha),x\rangle=\frac{1}{2}\omega(xv,v)=\frac{1}{2}\langle xv,\alpha\rangle-\frac{1}{2}\langle v,x\alpha\rangle=\langle xv,\alpha\rangle \quad \square$$

We will need the following choice of (G,V_0). Fix $n\in\mathbb{Z}_{>0}$. Let U be an n-dimensional space. Set $G=GL(U)$ & let $V=\operatorname{End}(U)\oplus U$ w. a natural G-action. We identify $\operatorname{End}(U)$ w. $\operatorname{End}(U)^*$ (& $\mathfrak{g}$ w. $\mathfrak{g}^*$ using the trace form: $(A,B):=\operatorname{tr}(AB)$. So we can identify V w.

$$V=\operatorname{End}(U)^{\oplus 2}\oplus U\oplus U^*$$

& view μ as a morphism $V \longrightarrow End(U)$.

Lemma 2: Under these identifications, we have

$$\mu(A,B,i,j) = [A,B] + ij \ , \quad A,B \in End(U), i \in U, j \in U^*,$$

where we view i as a linear map $\mathbb{C} \to U$ so that ij is a rk 1 linear operator $U \to U$, $u \mapsto j(u)i$.

Proof: Let $\mu_1: End(U)^{\oplus 2} \longrightarrow End(U)$, $\mu_2: U \oplus U^* \longrightarrow End(U)$ be the moment maps for the actions of $GL(U)$. By properties of moment maps explained in Sec 1.3 of Lec 16, we have

$$\mu(A,B,i,j) = \mu_1(A,B) + \mu_2(i,j).$$

We have $tr(\mu_2(i,j)x) = \langle j, xi \rangle = tr(jxi) = tr(ijx)$.

Since the trace form is non-degenerate, this implies $\mu_2(i,j) = ij$.

Similarly, $tr(\mu_1(A,B)x) = tr(B(x.A)) = [x.A = [x,A]] = tr(B[x,A])$

$= tr([A,B]x) \Rightarrow \mu_1(A,B) = [A,B]$. $\square$

1.2) Semistable points.

The lattice $\mathfrak{X}(G)$ is identified with $\mathbb{Z}$ via $n \mapsto det^n$. Similarly, $(\mathfrak{g}^*)^G$ is identified w. $\mathbb{C}$ via $z \mapsto z\, tr$.

We start by describing $V^{\theta-ss}$ for $\theta > 0$ and $\theta < 0$. Recall, Sec 1.2 of Lec 17, that $V^{\theta-ss}$ depends only on the sign of θ.

Proposition:

1) For $\theta<0$, $(A,B,i,j)\in V^{\theta\text{-ss}} \Leftrightarrow$ i is "cyclic" for A,B, i.e. U is the only A & B-stable subspace containing i is U.

2) For $\theta>0$, $(A,B,i,j)\in V^{\theta\text{-ss}} \Leftrightarrow$ j is "cocyclic" for A,B, i.e. $\{0\}$ is the only A & B-stable subspace contained in $\ker j$.

Proof: 1) We use the Hilbert-Mumford criterium for semistability (1) of Thm in Sec 1.1 of Lec 18): $(A,B,i,j)\in V^{\theta\text{-ss}}$ if from the existence of $\lim_{t\to 0}\gamma(t).(A,B,i,j)$ for $\gamma:\mathbb{C}^\times\to G$ it follows that $\langle\theta,\gamma\rangle\leq 0$. We write $U=\bigoplus_{n\in\mathbb{Z}} U_n(\gamma)$ w. $U_n(\gamma)=\{u\in U \mid \gamma(t)u=t^n u\}$. Then:

- $\lim_{t\to 0}\gamma(t)A$ exists iff $\bigoplus_{n\geq m} U_n(\gamma)$ is A-stable for all $m\in\mathbb{Z}$.

Same for B.

- $\lim_{t\to 0}\gamma(t)i$ exists iff $i\in\bigoplus_{n\geq 0} U_n(\gamma)$.
- $\lim_{t\to 0}\gamma(t)j$ exists iff j vanishes on $\bigoplus_{n>0} U_n(\gamma)$.

So if i is cyclic then $U=\bigoplus_{n\geq 0} U_n(\gamma)$. Since

$$\langle\gamma,\theta\rangle=\theta\sum_n n\dim U_n(\gamma).$$

we see that if $\lim_{t\to 0}\gamma(t)(A,B,i,j)$ exists, then $\langle\gamma,\theta\rangle\leq 0$, so $(A,B,i,j)\in V^{\theta\text{-ss}}$. Conversely, let i be not cyclic. Then let $U'\subsetneq U$ be an (A,B)-stable subspace containing i & let U'' be a complement. Take γ w. $U_0(\gamma)=U'$, $U_{-1}(\gamma)=U''$. For this γ, we have $\langle\gamma,\theta\rangle=-\theta\dim U''>0$, showing $(A,B,i,j) \notin V^{\theta\text{-}ss}$.

2) The case of $\theta > 0$ is similar & is left as an exercise.
Here's a stronger statement

Exercise: Identify U w. U^* yielding $A^*, B^* \in \mathrm{End}(U)$, $j^*: \mathbb{C} \to U$ & $i^*: U \to \mathbb{C}$. Then $(A,B,i,j) \mapsto (B^*, A^*, j^*, i^*)$ restricts to

$$V^{\theta\text{-}ss} \xrightarrow{\sim} V^{(-\theta)\text{-}ss} \qquad \square$$

Corollary: For $\theta \neq 0$, $G \curvearrowright V^{\theta\text{-}ss}$ is free.

Proof:

For $\theta < 0$ use that any element of U can be written as $P(A,B)i$, where P is a noncommutative polynomial in 2 variables. An element of G that stabilizes A, B, i also stabilizes $P(A,B)i$, hence is trivial.

The case of $\theta > 0$ is handled using Exercise in the proof of Proposition. $\square$

1.3) Calogero-Moser spaces. ([E], Sec 1.5)

Assume in this section that $\lambda \neq 0$.

Proposition: Let $(A,B,i,j)\in\mu^{-1}(\lambda)$. Then

(a) The only A & B-stable subspaces in U are U & $\{0\}$

(b) The stabilizer of (A,B,i,j) in G is trivial.

(c) $\mu^{-1}(\lambda)^{\theta\text{-ss}}=\mu^{-1}(\lambda)$.

Proof:

(a) Let $U'\subsetneq U$ be a nonzero (A,B)-stable subspace. Let $C:=[A,B]-\lambda\,\mathrm{Id}$, this is a rk 1 operator, equal to $-ij$. We write A',B',C' for the restrictions of A,B,C to U' & A'',B'',C'' for the induced operators on $U'':=U/U'$. We have $\mathrm{tr}\,C'=\mathrm{tr}\,[A',B']-\mathrm{tr}\,\lambda\,\mathrm{Id}_{U'}=-\lambda\dim U'$ $\Rightarrow C'\neq 0$ & similarly $C''\neq 0$. This implies $\mathrm{rk}\,C\geqslant \mathrm{rk}\,C'+\mathrm{rk}\,C''\geqslant 2$, contradiction, proving (a).

(b) By Schur's Lemma applied to (a), we see that the stabilizer consists of scalar operators. Note that $i,j\neq 0$, so if a scalar operator acts trivially on i it must be the identity

(c) is a direct corollary of (a) & Proposition in Sec 1.2. $\square$

Corollary: 1) $\mu^{-1}(\lambda)$ is a smooth subvariety in V of pure codimension n^2

2) $V/\!\!/\!\!/^{\theta}_{\lambda}G$ is a smooth affine symplectic variety of $\dim=2n$ independent of θ.

Proof: left as an exercise: use the previous proposition. Use

Sec 1.1 of Lec 16 for 1). Use Corollary in Sec 1.4 of Lec 14 to show that $\mu^{-1}(\lambda) \to \mu^{-1}(\lambda)/\!/G$ is a principal G-bundle; Sec 1.0 to establish an algebraic symplectic form; and Proposition in Sec 1.3 of Lec 17 to show that

$$V/\!/\!/_\lambda^\theta G \longrightarrow V/\!/\!/_\lambda G$$

is an isomorphism. $\square$

Remark: The reduction $V/\!/\!/_\lambda^\theta G$ is known as the Calogero-Moser space, it is a compactified phase space of the Calogero-Moser system and was defined by Kazhdan, Kostant & Sternberg.

1.4) $Hilb_n(\mathbb{C}^2)$ ([Nak], Sec 1)

Here we investigate $\mu^{-1}(0)/\!/^\theta G$ w. nonzero θ. Here's the main technical result.

Proposition: Let $(A,B,i,j) \in \mu^{-1}(0)^{\theta-ss}$.

1) If $\theta<0$, then $j=0$
2) If $\theta>0$, then $i=0$.

Before proving this result, let's explain it's significance. Assume $\theta<0$ first. Since $j=0$, we have $ij=0 \Rightarrow [A,B]=0$. So $\mu^{-1}(0)^{\theta-ss}=$

$=\{(A,B,i,0) \mid \mathbb{C}[A,B]i=\mathbb{C}^n\}$. The action of G on $\mu^{-1}(0)^{\theta\text{-}ss}$ is free. Then $\mu^{-1}(0)/\!\!/^{\theta}G \xrightarrow{\sim}$ the set of orbits $\mu^{-1}(0)^{\theta\text{-}ss}/G$ (recall that $\mu^{-1}(0)/\!\!/^{\theta}G$ parameterizes closed G-orbits in $\mu^{-1}(0)^{\theta\text{-}ss}$).

Note that $\{f\in\mathbb{C}[x,y] \mid f(A,B)i=0\}$ is a codimension n ideal in $\mathbb{C}[x,y]$ depending only on the G-orbit of (A,B,i). Conversely, given a codimension n ideal $I\subset\mathbb{C}[x,y]$, we can choose an identification $\mathbb{C}[x,y]/I\xrightarrow{\sim}U$ & define $A,B\in End(U)$ as the operators of multiplication by x & y & i as $1+I\subset\mathbb{C}[x,y]$. Then $(A,B,i,0)\in\mu^{-1}(0)^{\theta\text{-}ss}$. Note that different identifications $\mathbb{C}[x,y]/I\xrightarrow{\sim}U$ differ by an element of G and so we get maps between $\mu^{-1}(0)/\!\!/^{\theta}G$ & $\{$codim n ideals in $\mathbb{C}[x,y]\}$. In fact, these maps are mutually inverse (exercise). A stronger statement is true: $\mu^{-1}(0)/\!\!/^{\theta}G$ is a "fine moduli space" (it comes w. a universal family) parameterizing codim n ideals known as the Hilbert scheme $Hilb_n(\mathbb{C}^2)$.

Similarly to Corollary in Sec 1.3 the Hamiltonian reduction construction shows that $Hilb_n(\mathbb{C}^2)$ is a smooth symplectic variety of dimension $2n$.

The case of $\theta>0$ is handled using Exercise in Sec 1.2 (left as exercise).

Proof of Proposition:

Step 1: We start w. the following linear algebra fact: if $A,B\in \mathrm{End}(U)$ satisfy $\mathrm{rk}[A,B]\leq 1$, then A,B are upper-triangular in some basis. It's enough to show that U admits a proper $A\&B$-stable subspace, then we can argue by induction on $\dim U$.

We can replace A w. $A-\lambda\cdot \mathrm{Id}_U$ for suitable λ & assume A is degenerate. If $\ker A$ is B-stable, then we are done. Otherwise $\exists\, u_0\in\ker A$ w. $v:=ABu_0\neq 0$. Note that $v=[A,B]u$ & then $\mathrm{im}[A,B]=\mathbb{C}v$. We claim that $\mathrm{Im}\, A$ is B-stable. Indeed, $BAu=ABu-[A,B]u$. The 1st summand is in $\mathrm{Im}\, A$ & the 2nd is in $\mathbb{C}v\subset \mathrm{Im}\, A$. So $\ker A$ or $\mathrm{Im}\, A$ are a proper A-stable subspace.

Step 2: We claim that for $(A,B,i,j)\in\mu^{-1}(0)$ we have

(*) $$\langle j, \mathbb{C}\langle A,B\rangle i\rangle=0,$$

where we write $\mathbb{C}\langle A,B\rangle$ for the algebra of noncommutative polynomials in $A\&B$; (*) & Proposition in Sec 1.2 finish the proof.

To prove (*) we note that

$\langle j, f(A,B)i\rangle = tr(f(A,B)ij) = -tr(f(A,B)[A,B]) =$ [A,B are upper triangular in some basis $\Rightarrow [A,B]$ is strictly upper triangular, so is $f(A,B)[A,B]$] $=0$ $\square$

1.5) Remarks

1) One can also describe $\mu^{-1}(0)/\!/G$ although this is technically the hardest ([GG], Sec 2).

$$(**) \qquad (\mathbb{C}^2)^n/S_n \longrightarrow \mu^{-1}(0)/\!/G$$

is an isomorphism, where the morphism is induced by

$$(x_1,\dots x_n; y_1,\dots,y_n) \mapsto (\mathrm{diag}(x_1,\dots x_n), \mathrm{diag}(y_1,\dots y_n), 0, 0).$$

One can show that:

- $(**)$ is surjective on the level of points – using Step 1 of the proof of Proposition in Sec 1.4.
- $(**)$ is a closed embedding: by using a description of generators of $\mathbb{C}[(\mathbb{C}^2)^n]^{S_n}$ going back to Weyl.
- $\mu^{-1}(0)$ (and hence $\mu^{-1}(0)/\!/G$) is reduced. This requires:
  - showing that $\mu^{-1}(0)$ has $n+1$ irreducible components characterized by $\dim \mathbb{C}\langle A,B\rangle i$ at the generic point.
  - showing that each of the components contains a free orbit
  - using properties of moment maps to show that $\mu^{-1}(0)$ is generically reduced of $\dim = 2n^2+n$ and then deducing it's reduced.

We also note that the natural morphism $\mu^{-1}(0)/\!/^\theta G \to \mu^{-1}(0)/\!/G$ (for $\theta<0$) is the Hilbert-Chow map sending an ideal I to its support counted w. multiplicities.

2) This construction generalizes to $PGL(v) \curvearrowright Rep(Q,v)$, see Example 2 in Sec 2 of Lec 18 (the current construction is a special case: $\circlearrowleft \underset{n}{\circ} \leftarrow \underset{1}{\circ}$). The resulting Hamiltonian reductions are known as Nakajima quiver varieties and are very important in Geometric representation theory & Mathematical Physics.