Invariant theory 2, 01/15/25

1) Averaging operators & applications

Ref: [PV], Sec 3.4.

1) Averaging operators & applications

Let $\mathbb{F}$ be a field, V be a finite dimensional vector space/$\mathbb{F}$. We write $\mathbb{F}[V]$ for the algebra of polynomial functions, i.e. the symmetric algebra of V^*, denoted by $S(V^*)$ (it embeds into functions $V \to \mathbb{F}$ if $\mathbb{F}$ is infinite).

Let G be a group equipped with a homomorphism $G \to GL(V)$. In particular, G acts on $\mathbb{F}[V]$ by algebra automorphisms & we can form the subalgebra of invariants $\mathbb{F}[V]^G$.

Question: When is $\mathbb{F}[V]^G$ finitely generated?

Our goal in this lecture is to find sufficient conditions for affirmative answer.

1.1) Averaging operators, axiomatically.

When we work w. representations, V, of a finite group G, it's useful to consider the averaging operator

$$\alpha_V : V \longrightarrow V, \ v \mapsto \frac{1}{|G|} \sum_{g \in G} gv$$

that makes sense when char $\mathbb{F}$ doesn't divide $|G|$. We want to axiomatize some of its properties.

Definition (uncommon): Let G be a group. By a class of finite dimensional representations of G we mean a set of G-representations (up to isomorphism) that is closed under:

direct sums, tensor products, duals, taking subs & quotients

& contains the trivial representation.

An example is provided by all finite dimensional representations of a finite group G. We'll see other examples later.

Definition: Let $\mathcal{C}$ be a class of representations. By an averaging operator α for $\mathcal{C}$ we mean a collection of linear operators $\alpha_V : V \longrightarrow V$ $(V \in \mathcal{C})$ s.t.:

(a1) $\operatorname{im} \alpha_V \subset V^G = \{v \in V \mid gv = v \ \forall g \in G\}$

(a2) $\alpha_V(v) = v \ \forall v \in V^G$

(a3) $\forall$ G-equivariant linear maps $\varphi : U \to V$ $(U, V \in \mathcal{C})$ have $\varphi \circ \alpha_U = \alpha_V \circ \varphi$.

In other words, α is a functorial projector to the subspaces of invariants.

An example is provided by the averaging operator $\alpha = \frac{1}{|G|}\sum_{g\in G} g$ for finite G (& $\operatorname{char}\mathbb{F} \nmid |G|$) considered above.

Rem: An educated name for a "class" is "rigid monoidal full abelian subcategory of the category of finite dimensional representations of G."

1.2) Hilbert's finite generation theorem.

Thm: Let $\mathcal{C}$ be a class of representations that has averaging operator. Let $V\in\mathcal{C}$. Then $\mathbb{F}[V]^G$ is finitely generated.

Scheme of proof:

1) We'll reduce the question to the finite generation of a suitable ideal $I\subset\mathbb{F}[V]^G$. Note that $\tilde{I} = \operatorname{Span}_{\mathbb{F}[V]}(I)$ (the ideal in $\mathbb{F}[V]$ generated by I) is finitely generated by the Hilbert basis theorem

2) This is the main part: we will use the averaging operator to deduce that I is finitely generated from $\tilde{I}$ being finitely generated.

Proof: Set $\tilde{A} := \mathbb{F}[V] \supset A := \mathbb{F}[V]^G$

Step 1 (positively graded algebras) Note that $\tilde{A}$ is graded by $\mathbb{Z}_{\geq 0}$: $\tilde{A} = \bigoplus_{i \geq 0} \tilde{A}_i$, where $\tilde{A}_i$ $(= S^i(V^*))$ is the space of homogeneous deg i polynomials (being an "algebra grading" means $1 \in \tilde{A}_0$ & $\tilde{A}_i \tilde{A}_j \subset \tilde{A}_{i+j}$ $\forall\, i,j$). This grading is preserved by the G-action (for example, if we choose a basis then every $g \in G$ acts by linear changes of coordinates – or we can see this directly from the action on $S(V^*)$). It follows that A is a graded subalgebra, i.e. $A = \bigoplus_{i \geq 0} A_i$ w. $A_i := \tilde{A}_i \cap A$. Note that $A_0 = \tilde{A}_0 = \mathbb{F}$.

Step 2: Now let A be any $\mathbb{Z}_{\geq 0}$-graded commutative algebra w. $A_0 = \mathbb{F}$. Set $A_{>0} := \bigoplus_{i>0} A_i$, this is an ideal in A.

Exercise: Let $a_1, \dots a_k \in A_{>0}$ be homogeneous elements. TFAE:

(1) $a_1, \dots a_k$ generate A as an algebra.

(2) $a_1, \dots a_k$ generate $A_{>0}$ as an ideal.

In particular, A is finitely generated (as an algebra) iff $A_{>0}$ is finitely generated (as an ideal).

Set $I := A_{>0}$ & $\tilde{I} := \mathrm{Span}_{\tilde{A}}(I)$. We want to show that I is finitely generated. By the Hilbert basis thm, $\tilde{A}$ is Noetherian, so in any collection of generators of $\tilde{I}$ we can choose finite set of generators. In particular $\exists\, f_1,\dots f_k \in I$ generating $\tilde{I}$. We claim that $f_1,\dots,f_k$ generate I.

Step 3: Here we define the averaging operator $\tilde{\alpha}: \tilde{A} \to \tilde{A}$. Set $\tilde{A}_{\leq i} = \bigoplus_{j=0}^{i} \tilde{A}_i$ for $i \in \mathbb{Z}_{\geq 0}$ so that each $\tilde{A}_{\leq i}$ is G-stable & $\tilde{A} = \bigcup_{i \geq 0} \tilde{A}_{\leq i}$. We claim that $\tilde{A}_{\leq i} \in \mathcal{C}$. For this, note that:

- $V^* \in \mathcal{C}$ ($\mathcal{C}$ is closed under duals)
- $(V^*)^{\otimes i} \in \mathcal{C}$ ($\mathcal{C}$ is closed under $\otimes$)
- $S^i(V^*) \in \mathcal{C}$ (b/c it's a quotient of $(V^*)^{\otimes i}$)
- $\tilde{A}_{\leq i} \in \mathcal{C}$ (b/c $\mathcal{C}$ is closed under $\oplus$)

Let $\alpha_{\leq i}: \tilde{A}_{\leq i} \to \tilde{A}_{\leq i}$ be the averaging operator on $\tilde{A}_{\leq i}$. Let $j > i$ & $\iota: \tilde{A}_{\leq i} \hookrightarrow \tilde{A}_{\leq j}$ be the inclusion, it's G-equivariant. So, by condition (a3) in Sec 1.1, $\iota \circ \alpha_{\leq i} = \alpha_{\leq j} \circ \iota$. It follows that the following gives a well-defined operator $\tilde{A} \to \tilde{A}$:

$\tilde{\alpha}(a) := \alpha_{\leq i}(a)$ if $a \in \tilde{A}_{\leq i}$.

Step 4: Let $h \in \tilde{A}$, $f \in A = \tilde{A}^G$. We claim $\tilde{\alpha}(fh) = f\tilde{\alpha}(h)$. Indeed, assume $h \in \tilde{A}_{\leq i}$, $f \in A_{\leq j}$. We have a linear map $\varphi: \tilde{A}_{\leq i} \to \tilde{A}_{\leq i+j}$, $\varphi(a) := fa$. It's G-equivariant: $g.(fa) = (g.f)(g.a) = f(g.a)$. Again, by (a3), $\alpha_{\leq i+j} \circ \varphi = \varphi \circ \alpha_{\leq i} \Rightarrow \tilde{\alpha}(fh) = f\tilde{\alpha}(h)$.

Step 5: Now we prove the claim in the end of Step 2: $f_1, \ldots f_k$ generate I. Pick $F \in I$. Since $f_1, \ldots f_k$ generate $\tilde{I}$ $\exists\, h_1, \ldots, h_k \in \tilde{A}$ $F = \sum_{i=1}^{k} h_i f_i \Rightarrow F = \tilde{\alpha}(F) = [\text{Step } 4] = \sum \tilde{\alpha}(h_i) f_i$ & $\tilde{\alpha}(h_i) \in A$. $\square$

Rem: In particular, if G is finite & char $\mathbb{F}$ doesn't divide $|G|$, then $\mathbb{F}[V]^G$ is finitely generated. In fact (at least when char $\mathbb{F} = 0$) there's a stronger & more elementary result due to Noether: $\mathbb{F}[V]^G$ is generated by elements of deg $\leq |G|$.

1.3) Other examples of averaging operators.

Of course, the real usefulness of Hilbert's theorem is that averaging operators exist beyond the setting of finite groups. Below in this section $\mathbb{F} = \mathbb{C}$ for simplicity.

Case 1: K is a compact Lie group (e.g. $U(n)$ or $O_n(\mathbb{R})$). Let $\mathfrak{k} = \text{Lie}(K)$, $n = \dim K$.

Note that $\Lambda^n \mathfrak{k}^*$ is isomorphic to the space of left K-invariant top forms on K (via taking the fiber at 1). So $\exists!$ left invariant top form, say ω, s.t. $\int_K \omega = 1$ (it's here that we use that K is compact).

Now if f is a C^∞-function on K or, more generally a C^∞-map $K \longrightarrow V$, where V is a finite dimensional $\mathbb{C}$-vector space, we can consider $\int_K f\omega \in V$. Here are two important properties:

(P1) For $h \in K$, let $f^h(k) = f(hk)$. Then $\int_K f^h\omega = \int_K f\omega$

(P2) Let $\varphi : V \to V'$ be a linear map. Then $\int_K [\varphi \circ f]\omega = \varphi\left(\int_K f\omega\right)$. This is because taking the integral is linear in the integrand.

Now let V be a C^∞-representation of K, i.e. the matrix coefficients are C^∞-functions. Equivalently, $\forall v \in V$, the function $f_v : K \longrightarrow V$, $k \mapsto k.v$, is C^∞. Set:

$$\alpha_V(v) = \int_K f_v \omega.$$

Note that the C^∞-representations of K form a class in the sense Sec 1.1.

Proposition 1: The collection $\alpha = (\alpha_V)$ for the class of C^∞-representations is an averaging operator.

Proof: (a3) is a direct consequence of (P2) & (a2) is left as an exercise (hint: $f_v = v$ $\forall\, v \in V^K$). Let's check (a1):

$$h\alpha_V(v) = h\int_K f_v\,\omega = [\text{P2 applied to } h: V \to V] = \int_K hkv\,\omega =$$
$$= \int_K (f_v)^h\,\omega = [\text{P1}] = \int_K f_v\,\omega = \alpha_V(v)$$ $\square$

Case 2: G is an (affine) algebraic group, i.e. an affine variety – recall that the base field is $\mathbb{C}$ – equipped w. morphisms $m: G \times G \to G$ (multiplication) and $i: G \to G$ (inverse) making G into a group.

Examples: $GL_n(\mathbb{C})$ & its Zariski closed subgroups $SL_n(\mathbb{C})$, $O_n(\mathbb{C})$, $Sp_n(\mathbb{C})$ (for even n), are algebraic.

It makes sense to talk about algebraic group homomorphisms (= morphisms of varieties that are group homomorphisms). Also, every algebraic group is a complex Lie group, so it makes to speak about their Lie algebras.

Now we are going to introduce a class of representations that we will consider

Def: Let V be a finite dimensional vector space. By a rational representation of G in V we mean a representation given by an algebraic group homomorphism.

The terminology is explained as follows: for $G = GL_n(\mathbb{C})$, the matrix coefficients are polynomial functions on G, in particular, are rational functions in the matrix entries.

The next exercise outlines basic properties of rational representations - and allows to construct a lot of them.

Exercise: 1) Rational representations of G form a class in the sense of Sec 1.1. Denote it by $\mathcal{C}$.

2) If $H \subset G$ is an algebraic subgroup & V is a rational representation of G, then it's also rational representation of H.

3) The tautological representation of $GL_n(\mathbb{C})$ is rational.

It's not true that the averaging operator for $\mathcal{C}$ exists for any algebraic group G. Here's a sufficient (and, as we will mention later, necessary) condition.

Definition: G is reductive if it contains a Zariski dense compact Lie subgroup

Lemma: Assume G connected in the usual topology (in fact, this is equivalent to G being irreducible see [OV], § 3.3.1). Let $K \subset G$ be a compact Lie subgroup s.t. $\mathfrak{k} \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{g}$. Then K is Zariski dense in G, hence G is reductive.

Proof: Let $G' \subset G$ be the Zariski closure of K, an algebraic subgroup of G. We have $\mathfrak{k} \subset \mathfrak{g}'$ & $\mathfrak{g}' \subset \mathfrak{g}$ is a $\mathbb{C}$-subspace $\Rightarrow$ $\mathfrak{g}' = \mathfrak{g} \Rightarrow$ [G is connected] $G' = G$. $\square$

Examples:
- $G = GL_n(\mathbb{C}) \supset K = U(n)$,
- $G = SO_n(\mathbb{C}) \supset K = SO_n(\mathbb{R})$,
- $G = Sp_{2n}(\mathbb{C}) \supset K = Sp_n$ = unitary transformations of $\mathbb{H}^n$, where $\mathbb{H}$ is the quaternions.

A rational representation V of G restricts to a C^∞-representation of K. For $v \in V$, let $f_v : K \to V$, $k \mapsto kv$. Set $\alpha_V(v) = \int_K f_v \omega$.

Proposition 2: $\alpha = (\alpha_V)$ is an averaging operator for the rational representations of G.

Proof: (a2) & (a3) follow from Proposition 1. (a1) reduces to

(*) $\qquad V^K = V^G.$

Let $v \in V^K$. Note that $\mathrm{Stab}_G(v) \subset G$ is Zariski closed. Since K is Zariski dense in G, we get $\mathrm{Stab}_G(v) = G \Rightarrow v \in V^G \Rightarrow$ (*). $\square$