Lecture 20, 4/2/2025.

1) Kempf-Ness theorem for GIT quotients
2) Hyper-Kähler reductions

Refs: see below.

1) Kempf-Ness theorem for GIT quotients (Ref: [Ki], Sec 6)

First, we need to recall a few things from Lec 15. Let G be a reductive group with maximal compact subgroup $K \subset G$. Let G act on a finite dimensional rational representation V & let $(\cdot,\cdot)$ be a K-invariant Hermitian scalar product; unlike in the previous lectures is assumed to be linear in the 2nd argument. This gives rise to a K-invariant symplectic form $\omega = -2\,\mathrm{Im}(\cdot,\cdot)$ & a moment map $\mu: V \to \mathfrak{k}^*$, $\langle \mu(v), x\rangle = \sqrt{-1}(xv, v) = \frac{1}{2}\omega(xv,v)$ making the K-action on V Hamiltonian.

Recall that the Kempf-Ness theorem proved in Lec 15 states that $Gv \cap \mu^{-1}(0) \neq \varnothing$ iff Gv is closed and if so the intersection is a single K-orbit. The goal of this section is to state & prove an analog of the theorem for GIT quotients. Pick $\theta \in \mathfrak{X}(G)$.

Identifying θ w. its differential we view θ as a homomorphism $\mathfrak{g} \to \mathbb{C}$. It maps $\mathfrak{k}$ to $\sqrt{-1}\mathbb{R}$. So $\sqrt{-1}\theta$ can be viewed as an element of $\mathfrak{k}^*$. Note that it's K-invariant.

Theorem (King) Let $v \in V^{\theta-ss}$. Then Gv is closed in $V^{\theta-ss}$ iff $Gv \cap \mu^{-1}(\sqrt{-1}\theta) \neq \emptyset$. If this intersection is nonempty, then it's a single K-orbit.

We start w. a basic & classical example

Example: Consider $V = \mathbb{C}^n$ w. the usual Hermitian scalar product & $G = \mathbb{C}^\times$ acting by inverse scaling action $t.v = t^{-1}v$, so that $\mathbb{C}[V]$ is positively graded. Take $\theta = 1$. The moment map μ is given by $(z_1, \dots z_n) \mapsto \sqrt{-1}\sum_{i=1}^n |z_i|^2$. The preimage of $\sqrt{-1}\theta$ is $\{z \mid |z|^2 = 1\}$. Clearly, the intersection of $\mu^{-1}(\sqrt{-1}\theta)$ with every (automatically closed) $\mathbb{C}^\times$-orbit is an $\mathbb{S}^1$-orbit.

Partial proof: We prove that if $Gv \cap \mu^{-1}(\sqrt{-1}a\theta) \neq \emptyset$, then it's a single K-orbit & Gv is closed in $V^{\theta-ss}$. Consider the G-action of $V \oplus \mathbb{C}_\theta$ & recall (Exercise 2 in Sec 1.1 of Lec 18) that $Gv \subset V^{\theta-ss}$ is closed $\Leftrightarrow$ so is $G.(v,1) \subset V \oplus \mathbb{C}_\theta$. Equip $V \oplus \mathbb{C}_\theta$ w. Hermitian form

$$((v_1, z_1), (v_2, z_2)) = (v_1, v_2) + \bar{z}_1 z_2.$$

The corresponding moment map $\tilde{\mu}: V \oplus \mathbb{C}_\theta \to \mathfrak{k}^*$ is given by

$$\tilde{\mu}(v, z) = \mu(v) - \sqrt{-1}|z|^2\theta$$

In particular, if $\mu(v) = \sqrt{-1}\,\theta$, then $(v,1) \in \tilde{\mu}^{-1}(0)$. The Kempf-Ness theorem applied to $G \curvearrowright V \oplus \mathbb{C}_\theta$ shows $G(v,1)$ is closed & $\tilde{\mu}^{-1}(0) \cap G(v,1) = K(v,1) \Rightarrow$ [exercise] $\mu^{-1}(\sqrt{-1}\,\theta) \cap Gv = Kv$.

Note that this also shows that any closed G-orbit in $V^{\theta\text{-}ss}$ intersects $\mu^{-1}(\mathbb{R}_{>0}\theta\sqrt{-1})$. $\square$

Exercise: Prove the theorem when $Z(GL(V)) \subset$ image of G in $GL(V)$.

Remark: Recall that $\mu_\theta := \mu - \sqrt{-1}\,\theta : V \to \mathfrak{k}^*$ is also a moment map. Let X be a closed G-stable affine subvariety in V. Assume that G acts freely on $X^{\theta\text{-}ss}$ & $X^{\theta\text{-}ss}$ is smooth. Then $X/\!/^\theta G$ is smooth (see a comment in Sec 1.0 of Lec 19). On the other hand, $X^{\theta\text{-}ss}$ is complex, hence symplectic, submanifold of V. The action $K \curvearrowright X^{\theta\text{-}ss}$ is Hamiltonian & free, so by Sec 1.2 in Lec 16, $(\mu_\theta^{-1}(0) \cap X^{\theta\text{-}ss})/K$ is smooth & symplectic. King's theorem says that the natural map $(\mu_\theta^{-1}(0) \cap X^{\theta\text{-}ss})/K \longrightarrow X/\!/^\theta G$ is a bijection. It can be shown to be a C^∞-isomorphism, in particular, equipping $X/\!/^\theta G$ with a C^∞ symplectic form. For instance, in Example we recover the Fubini-Study form on $\mathbb{CP}^{n-1}$ (a unique up to rescaling SU_n-invariant symplectic form on $\mathbb{CP}^{n-1}$).

2) Hyper-Kähler reductions (Ref [HKLR], [Pr])

For $\lambda \in (\mathfrak{g}^*)^G$, $\theta \in \mathfrak{X}(G)$ we can consider the GIT Hamiltonian reduction $V/\!\!/\!\!/_\lambda^\theta G = \mu^{-1}(\lambda)/\!\!/^\theta G$, a smooth symplectic variety in the case when G acts on $\mu^{-1}(\lambda)^{\theta-ss}$ freely.

In this section we will use the Kempf-Ness theorem perspective to observe an important symmetry relating different values on (λ, θ). It comes from the quaternionic structure on V.

2.0) Overview of quaternionic stuff.

Let $\mathbb{H}$ denote the skew-field of quaternions, its elements are of the form $h = a + bi + cj + dk$ w $i^2 = j^2 = k^2 = -1$, $ji = k$. Let $\bar{\cdot}: \mathbb{H} \to \mathbb{H}$ denote the quaternionic conjugation: $a + bi + cj + dk \mapsto a - bi - cj - dk$, so that $h\bar{h} = \bar{h}h = a^2 + b^2 + c^2 + d^2 =: |h|^2$; $\bar{\cdot}$ is an anti-involution.

A quaternionic vector space is a right $\mathbb{H}$-module. Every such module is free - by the same argument as for fields.

2.0.1) Hermitian forms.

Definition: Let V be a quaternionic vector space. By a quaternionic hermitian form on V we mean a map $(\cdot,\cdot)_{\mathbb{H}}: V \times V \longrightarrow \mathbb{H}$ satisfying $(v_2, v_1) = \overline{(v_1, v_2)}$, $(v_1, v_2 h) = (v_1, v_2)h$ $(v_i \in V, h \in \mathbb{H})$

Example: 1) $V=\mathbb{H}^n$ & $(\vec{x},\vec{y})=\sum_{i=1}^{n}\bar{x}_i y_i$ $\left(\vec{x}=(x_1,\dots x_n),\vec{y}=(y_1,\dots y_n)\right)$

2) We revisit 1) for $n=1$. Note that $\{a+bi\}\subset\mathbb{H}$ is a subfield identified w. $\mathbb{C}$. Then $\mathbb{H}=\mathbb{C}\oplus j\mathbb{C}$. We have

(1) $(z_1+jz_2)(w_1+jw_2)=(\bar{z}_1-\bar{z}_2 j)(w_1+jw_2)=[\,zj=j\bar{z}\,]=(\bar{z}_1w_1+\bar{z}_2w_2)+$
$+j(z_1w_2-z_2w_1)$.

As in the real & complex case quaternionic Hermitian forms are classified by their singnature as every form admits an orthogonal basis. We will be interested in positive definite forms (quaternionic scalar products), cf. Example.

We can write a quaternionic Hermitian form $(\cdot,\cdot)_{\mathbb{H}}$ as

$$(\cdot,\cdot)_{\mathbb{H}}=(\cdot,\cdot)_{\mathbb{R}}+i\omega_I+j\omega_J+k\omega_K,$$

where $(\cdot,\cdot)_{\mathbb{R}},\omega_I,\omega_J,\omega_K$ are $\mathbb{R}$-valued $\mathbb{R}$-bilinear forms on V.

Lemma 1: Assume $(\cdot,\cdot)_{\mathbb{H}}$ is a scalar product. Then

1) $(\cdot,\cdot)_{\mathbb{C}}=(\cdot,\cdot)_{\mathbb{R}}+i\omega_I(\cdot,\cdot)$ is a Hermitian scalar product $V\times V\to\mathbb{C}$, (where our convention is that it's $\mathbb{C}$-linear in the 2nd argument).

2) $\omega_I,\omega_J,\omega_K$ are real symplectic forms

3) $\omega:=\omega_J+i\omega_K$ is a complex symplectic form.

4) $(\cdot,\cdot)_{\mathbb{H}}=(\cdot,\cdot)_{\mathbb{C}}+j\omega$.

Sketch of proof: Use that $(\cdot,\cdot)_{\mathbb{H}}$ in some basis is given as in Example 1 & use the 2nd part of Example 1. $\square$

2.0.2) Symmetries.

Now let us discuss the symmetries. The automorphism group $GL_n(\mathbb{H})$ of $\mathbb{H}^n$ is the group of invertible quaternionic matrices. Its maximal compact subgroup, Sp_n, is the stabilizer of some positive definite quaternionic form $(\cdot,\cdot)_{\mathbb{H}}$. Note that this stabilizer is the intersection of the stabilizers of $(\cdot,\cdot)_{\mathbb{C}}$ (which is U_{2n}) & $\omega_{\mathbb{C}}$ (which is $Sp_{2n}(\mathbb{C})$ - note that Sp_n is a real form of $Sp_{2n}(\mathbb{C})$). So every compact Lie group acting on a complex symplectic vector space (V,ω) has an invariant quaternionic structure together with an invariant quaternionic scalar product $(\cdot,\cdot)_{\mathbb{H}}$.

2.1) Hyper-Kähler moment maps

Now suppose V is a quaternionic vector space w. quaternionic scalar product $(\cdot,\cdot)_{\mathbb{H}}$ & K be a compact Lie group acting by quaternionic transformations & preserving $(\cdot,\cdot)_{\mathbb{H}}$. Let $\omega_I, \omega_J, \omega_K, (\cdot,\cdot)_{\mathbb{C}}$ & ω have the same meaning as before. Let $\mu_I: V \to \mathfrak{k}^*$ denote the moment map for $2\omega_I$: $\langle \mu_I(v), x\rangle = \omega_I(xv, v)$. The purpose of the multiple is to make the exposition here compatible w. Sec 1.

Define μ_J, μ_K similarly. Set

$$\mu_{\mathbb{H}} = \mu_I i + \mu_J j + \mu_K k : V \longrightarrow \mathfrak{k}^* \otimes \{ai + bj + ck\}$$

We need the SU_2 symmetry property of $\mu_{\mathbb{H}}$. Note that SU_2 is identified w. $\{z \in \mathbb{H} \mid |z| = 1\}$ w. opposite multiplication (it acts on $\mathbb{H} = \mathbb{C}^2$ from the right). So we get the action of SU_2 on $\mathbb{H}$ by conjugation: it preserves $\{ai+bj+ck\}$ and acts on this space ($\mathbb{R}^3$) via $SU_2 \twoheadrightarrow SO_3$.

In particular, $SU_2 = \{z \in \mathbb{H} \mid |z| = 1\}^{opp}$ acts on V. It also acts on $\mathfrak{k}^* \otimes \mathbb{H}$ via conjugations on the 2nd factor.

Lemma: $\mu_{\mathbb{H}}$ is SU_2-equivariant.

Proof: We have $(xvh, vh)_{\mathbb{H}} =$ [$(\cdot,\cdot)_{\mathbb{H}}$ is quaternionic Hermitian] $\bar{h}(xv, v)_{\mathbb{H}} h = \bar{h}h(xv, v)_{\mathbb{R}} + \bar{h}\mu_{\mathbb{H}}(v)h$. Since $\bar{h}h \in \mathbb{R}$, we deduce that $\mu_{\mathbb{H}}(vh) = \bar{h}\mu_{\mathbb{H}}(v)h$ (as the imaginary components of $(xvh, vh)_{\mathbb{H}}$). $\square$

2.2) Reductions.

Now take $\alpha_I, \alpha_J, \alpha_K \in (\mathfrak{k}^*)^K$ and consider $\mu_{\mathbb{H}}^{-1}(\alpha_I i + \alpha_J j + \alpha_K k)$. Note that for $h \in SU_2$, the action of h on V gives rise to a K-equivariant isomorphism $v \mapsto vh : \mu_{\mathbb{H}}^{-1}(\alpha_I i + \alpha_J j + \alpha_K k) \xrightarrow{\sim} \mu_{\mathbb{H}}^{-1}(h^{-1}(\alpha_I i + \alpha_J j + \alpha_K k)h)$ and hence an isomorphism between their quotients by K.

Now we are going to interpret these quotients as GIT quotients (for certain choices of $\alpha_I, \alpha_J, \alpha_K$). Namely the complexification G of K acts on V (which is a $\mathbb{C}$-vector space) & ω is invariant. Let $\mu = \mu_J + \mu_K i : V \to \mathfrak{k}^* \oplus \mathfrak{k}^* i = \mathfrak{g}^*$ be the corresponding moment map.

Let $\theta_I, \theta_J, \theta_K : G \to \mathbb{C}^\times$ and let $\alpha_I = \sqrt{-1}\,\theta_I$ & similarly for α_J, α_K. Then by King's Theorem:

$$\mu_{\mathbb{H}}^{-1}(\alpha_I i + \alpha_J j + \alpha_K k)/K \xrightarrow{\sim} \mu^{-1}(\alpha_J + i\alpha_K)/\!/^{\theta_I} G.$$

Note that SU_2 acts transitively on the unit sphere in the imaginary quaternions. In particular, we can permute $\alpha_I, \alpha_J, \alpha_K$ getting isomorphisms (of C^∞-manifolds if the K-action on $\mu_{\mathbb{H}}^{-1}(\alpha_I i + \alpha_J j + \alpha_K k)$ is free (cf. Rem in Sec 2.2 of Lec 15)

(2) $$\mu^{-1}(\alpha_J + i\alpha_K)/\!/^{\theta_I} G \xrightarrow{\sim} \mu^{-1}(\alpha_I + i\alpha_K)/\!/^{\theta_J} G$$

This is an example of symmetry we are after.

Example: Back to the setting of Lec 19, let $CM_n := \mu^{-1}(\lambda)/\!/G$ ($\lambda \neq 0$) be the Calogero-Moser space (these spaces for different λ are isomorphic thx to the scaling $\mathbb{C}^\times$-action. (2) yields a C^∞-isomorphism $CM_n \xrightarrow{\sim} \mathrm{Hilb}_n(\mathbb{C}^2)$ (exercise).

Remarks: 1) Assume $G \curvearrowright \mu^{-1}(\lambda)^{\theta\text{-}ss}$ is free. The construction of this section equips (smooth) reduction

$$\mathcal{M} := \mu^{-1}(\lambda)/\!\!/^{\theta} G$$

with a so called "hyper-Kähler structure". By definition, this is a triple of Kähler structures $(I, \underline{\omega}_I), (J, \underline{\omega}_J), (K, \underline{\omega}_K)$ (where I, J, K are complex structures = endomorphisms of $T\mathcal{M}$ that square to $-Id$ & $\underline{\omega}_I, \underline{\omega}_J, \underline{\omega}_K$ are suitably compatible symplectic forms) s.t I, J, K satisfy, in addition $JI = K$ & $\underline{\omega}_I(\cdot, I\cdot) = \underline{\omega}_J(\cdot, J\cdot) = \underline{\omega}_K(\cdot, K\cdot)$ is a Riemannian metric. The holonomy group of this metric is contained in Sp_n, where $\dim \mathcal{M} = 2n$. Hyper-Kähler structures are hard to construct & reduction is a common way to do this.

2) We imposed integrality conditions of $\alpha_I, \alpha_J, \alpha_K$. This is not necessary: one can make sense of $X/\!\!/^{\theta} G$ for all $\theta \in \mathfrak{X}(G) \otimes_{\mathbb{Z}} \mathbb{R}$. We will address this in a future HW problem.