Lecture 21, 4/6/25

1) Parameterizing vector bundles on curves

2) Quot schemes

Ref: [Ne], Secs 1.1, 1.2, 5.1-5.3; [HL], Sec 2.2.

1) Parameterizing vector bundles on curves

Here C is a smooth projective curve over $\mathbb{C}$. We are interested in parameterizing (certain) vector bundles on C.

1.1) Poly/semi/stable vector bundles.

We write $K_0(C)$ for the Grothendieck group of coherent sheaves on C. For $\mathcal{F} \in Coh(C)$ we write $[\mathcal{F}]$ for its class in $K_0(C)$. We consider two group homomorphisms $K_0(C) \to \mathbb{Z}$:

i) rk (rank) sending $[\mathcal{F}]$ to the dimension of the fiber of $\mathcal{F}$ at the generic point, Spec $\mathbb{C}(X)$.

ii) deg (degree), a unique homomorphism sending $[\mathcal{O}_C]$ to 0 & every skyscraper $[\mathbb{C}_x]$ $(x \in C)$ to 1.

Definition: For $\mathcal{F} \in Coh(X)$ define the slope
$$\mu(\mathcal{F}) := \frac{\mathrm{rk}(\mathcal{F})}{\deg(\mathcal{F})} \in \mathbb{Q} \sqcup \{+\infty\}$$
Note that $\mu(\mathcal{F}) = +\infty \Leftrightarrow \mathrm{rk}(\mathcal{F}) = 0 \Leftrightarrow \mathcal{F}$ is torsion.

Definition: We say that $\mathcal{F}$ is

- **semistable**: if $\forall\, \mathcal{F}' \subsetneq \mathcal{F}\ (\mathcal{F}' \neq 0) \Rightarrow \mu(\mathcal{F}') \leq \mu(\mathcal{F})$
- **stable**: — · — · — · — · — · — · — · — · $<$ — · —
- **polystable**: if $\mathcal{F}$ is isomorphic to the direct sum of stable sheaves of slope $\mu(\mathcal{F})$.

Examples/exercises:

0) A torsion sheaf is always semistable. It's stable iff it's $\mathbb{C}_x$ for some $x \in C$.

1) Any semistable sheaf of slope $< +\infty$ is a vector bundle

2) Any line bundle $\mathcal{L}$ is stable w. $\mu(\mathcal{L}) = \deg(\mathcal{L})$. Moreover $\mathcal{L} \otimes \cdot$ sends semi/poly/stable sheaves of slope μ to semi/poly/stable sheaves of slope $\mu + \mu(\mathcal{L})$

3) If $\mathcal{F}$ is semistable (resp. stable), then $\mu(\mathcal{F}'') \geq \mu(\mathcal{F})$ (resp. $\mu(\mathcal{F}'') > \mu(\mathcal{F})$) $\forall$ nonzero proper quotients $\mathcal{F}''$ of $\mathcal{F}$.

4) Let $\mathcal{F}_1, \mathcal{F}_2$ be semistable sheaves of slopes μ_1, μ_2. If $\mu_1 > \mu_2$, then $\mathrm{Hom}(\mathcal{F}_1, \mathcal{F}_2) = 0$. Moreover, if $\mu_1 = \mu_2$ & $\mathcal{F}_1, \mathcal{F}_2$ are stable, then any nonzero homomorphism $\mathcal{F}_1 \to \mathcal{F}_2$ is an isomorphism.

5) By definition, every semi-stable sheaf $\mathcal{F}$ admits a filtration $\{0\} = \mathcal{F}^0 \subsetneq \mathcal{F}^1 \subsetneq \mathcal{F}^2 \subsetneq \ldots \subsetneq \mathcal{F}^k = \mathcal{F}$ s.t. $\mathcal{F}^i/\mathcal{F}^{i-1}$ is stable of slope $\mu(\mathcal{F})$. The (polystable) sheaf $\mathrm{gr}\,\mathcal{F}$ is independent of

the choice of filtration.

6) If $GCD(r,d)=1$, then every semistable bundle is stable.

We seek to parameterize vector bundles on C up to isomorphism, roughly, by points of a variety. It's possible to do this for semistable bundles with an important caveat: $\forall$ semistable $\mathcal{F}$ goes to the same point as the polystable bundle $\operatorname{gr}\mathcal{F}$.

1.2) Fine & coarse moduli spaces.

A space parameterizing algebro-geometric objects is known as their moduli. Our description of this space (essentially as a set) was too naive (an analogy: to describe a GIT quotient as a set of closed orbits in the semistable locus). Here we define two rigorous notions: fine and coarse moduli space of stable bundles. The first is the strongest notion of a moduli space (but it rarely exists), the 2nd is relatively weak but always exists.

Definition: Fix $r\in\mathbb{Z}_{>0}$, $d\in\mathbb{Z}$. Consider the functor $Bun^s_{r,d}$ from (finite type) schemes to $Sets^{opp}$, $S\mapsto Bun^s_{r,d}(C\times S)=\{$rk r vector bundles $\mathcal{F}$ on $C\times S$ s.t. $\forall\, s\in S \Rightarrow \mathcal{F}|_{C\times\{s\}}$ is stable of deg $d\}/\sim$, where $\mathcal{F}\sim\mathcal{F}' \overset{def}{\Longleftrightarrow} \exists$ line bundle $\mathcal{L}$ on S w. $\mathcal{F}\xrightarrow{\sim}\mathcal{F}'\otimes pr^*(\mathcal{L})$.

A morphism $\varphi: S_1 \to S_2$ goes to the pullback $\mathcal{F} \mapsto \varphi^*\mathcal{F}$.

A fine moduli space is a representing object for this functor, i.e. a finite type scheme $\mathcal{M}^{s,f}_{r,d}$ together w. a vector bundle $\mathcal{F}^{un}$ on $C \times \mathcal{M}^{s,f}_{r,d}$ s.t. $\forall$ S & $\mathcal{F}$ as above $\exists!$ morphism $\varphi: S \to \mathcal{M}^{s,f}_{r,d}$ s.t. $\mathcal{F} \sim \varphi^*\mathcal{F}^{un}$. By Yoneda, $\mathcal{M}^{s,f}_{r,d}$ is unique if it exists.

Note that, by definition, the set of $\mathbb{C}$-points, $\mathcal{M}^{s,f}_{r,d}(\mathbb{C})$ is the set of isom. classes of stable rk r deg d vector bundles on C so we do get a scheme parameterizing such vector bundles.

A fine moduli space is something we ideally want to get (and when $GCD(r,d)=1$ actually get). A coarse moduli space, on the other hand is, in some rigorous way, the best we can get.

Definition: A (finite type) scheme $\mathcal{M}^{s,c}_{r,d}$ together w. a morphism $Bun^s_{r,d} \overset{\Phi}{\Rightarrow} Mor_{Sch}(\cdot, \mathcal{M}^{s,c}_{r,d})$ s.t.

1) $\forall$ (finite type) scheme M & $\Psi: Bun^{s,c}_{r,d} \Rightarrow Mor_{Sch}(\cdot, M)$ $\exists!$ $\varphi: \mathcal{M}^{s,c}_{r,d} \to M$ w. $\Psi = \varphi_* \circ \Phi$,

2) and $\Phi(pt): \{\text{stable bundles of deg } d, \text{ rk } r \text{ on } C\}/\sim \; \to \mathcal{M}^{s,c}_{r,d}(\mathbb{C})$ is a bijection

is called the coarse moduli space (of stable bundles). Again, $\mathcal{M}^{s,c}_{r,d}$ is unique if it exists.

Condition 1) is similar in spirit to the universality property of the categorical quotient, see Sec 1.3 of Lec 3. GIT quotients enjoy a similar universal property, & in fact, $\mathcal{M}^{s,c}_{r,\alpha}$ can be constructed as a GIT quotient.

1.3) Roadmap

Here's an approach, which works for vector bundles on a curve - and many other setups.

Step 1: we want to reduce the problem of constructing a moduli space to the problem of constructing a quotient for a reductive group action on a projective scheme. Roughly, there's an open locus in that scheme, where points correspond to vector bundles w. some additional structure & the group action correspond to changing that additional structure.

Step 2: There's a version of the GIT quotient construction for reductive group actions on projective schemes - which is basically a special case of what we have covered in Lecs 17, 18. One can apply this construction to parameterize certain orbits in the setting of Step 1 & one should relate the notions of semi/poly/stability in GIT (we've covered semistability & we'll get to poly/stability later) to those arising in Geometry.

2) Quot schemes

We proceed to Step 1 of the above plan.

2.1) Global generation & cohomology vanishing for s/stable bundles

Let ω_C denote the canonical bundle on C = the bundle of Kähler differentials. Recall that $\deg \omega_C = 2g-2$ where g is the genus of C.

Proposition: Let $r \in \mathbb{Z}_{>0}$. Let $\mathcal{F}$ be a semistable vector bundle on C of rank r & degree d. Then:

1) If $d > r(2g-2)$, then $H^1(\mathcal{F}) = 0$

2) If $d > r(2g-1)$, then $\mathcal{F}$ is generated by its global sections.

Proof:

1) Suppose $0 \neq H^1(\mathcal{F})^* =$ [Serre duality] $= H^0(\mathcal{F}^* \otimes \omega_C) = \mathrm{Hom}(\mathcal{F}, \omega_C)$. The condition $d > r(2g-2)$ means precisely that $\mu(\mathcal{F}) > \mu(\omega_C)$. Applying 4) from Sec 1.1, we get $\mathrm{Hom}(\mathcal{F}, \omega_C) = 0$.

2) Let $x \in C$, $\mathcal{F}_x$ is the fiber of $\mathcal{F}$ at x & $\mathfrak{m}_x \subset \mathcal{O}_C$ is the maximal ideal of x. We want to prove

$$H^0(\mathcal{F}) \twoheadrightarrow H^0(\mathcal{F}_x) = \mathcal{F}_x \quad \forall x.$$

Thx to SES $0 \to \mathfrak{m}_x \otimes \mathcal{F} \to \mathcal{F} \to \mathcal{F}_x \to 0$, it's enough to show $H^1(\mathfrak{m}_x \otimes \mathcal{F}) = 0$. Note that $\mathfrak{m}_x$ is a line bundle of $\deg = -1$

By 2) in Sec 1.1, $m_x \otimes \mathcal{F}$ is a semistable of slope $\mu(\mathcal{F})-1$, i.e. of degree $d-r$. We apply 1) & see $H^1(m_x \otimes \mathcal{F})=0$ $\square$

Remark: Note that we have a natural homomorphism

(*) $$H^0(\mathcal{F}) \otimes \mathcal{O}_C \to \mathcal{F}$$

inducing the identity homomorphism $H^0(\mathcal{F}) \xrightarrow{\sim} H^0(\mathcal{F})$. The claim that $\mathcal{F}$ is generated by its global sections means that (*) is onto.

2.2) Definition & basic properties

It turns out that there's a fine moduli space for quotients of a given coherent sheaf with a given discrete invariant, the Hilbert polynomial. This fine moduli space is known as the Quot scheme ("quot" from quotients). In this lecture we will state its existence and basic properties, we'll sketch a construction in a subsequent lecture.

First, let's recall the Hilbert polynomial. Let X be a projective scheme over $\mathbb{C}$ equipped w. a very ample line bundle $\mathcal{O}(1)$. For a coherent sheaf $\mathcal{F}$ on X, let $\mathcal{F}(m)$ denote $\mathcal{F} \otimes \mathcal{O}(1)^{\otimes m}$. Then the Euler characteristic

$$\chi(\mathcal{F}(m)) = \sum_{i \geq 0} (-1)^i \dim H^0(X, \mathcal{F}(m))$$

is known to be a polynomial in m, this is the Hilbert polynomial in question.

Example: Let $X = C$ and $\mathcal{F}$ be a vector bundle of degree d & rank r. Let d_0 denote the degree of $\mathcal{O}(1)$, it's positive. By Riemann-Roch theorem, $\chi(\mathcal{F}(m)) = \deg \mathcal{F}(m) + (1-g)\operatorname{rk}\mathcal{F}(m) = d + r\cdot d_0 m + (1-g)r$
$= (d + (1-g)r) + rd_0 \cdot m$.

Fix a nonzero vector space V and $P \in \mathbb{Q}[t]$. Consider the following functor from (finite type) schemes over $\mathbb{C}$ to Sets^{opp}: we send a test scheme S to the set of quotients $\mathcal{F}$ of $V \otimes \mathcal{O}_{X\times S}$ $(\simeq \mathcal{O}_{X\times S}^{\oplus \dim V})$ that are flat over S & s.t. $\chi(\mathcal{F}|_{X\times\{s\}}(m)) = P(m)$ for all points of S (note that the l.h.s is locally constant on S thx to the flatness assumption). To figure out what the functor does on morphisms is left as an exercise.

Denote the functor by $\underline{\mathrm{Quot}}_X^{P,V}$ (it also depends on the choice of $\mathcal{O}(1)$ but we suppress it from the notation). The following is due to Altman & Kleiman (a more general result is due to Grothendieck)

Proposition: The functor $\underline{\mathrm{Quot}}_X^{P,V}$ is represented by a projective scheme, $\mathrm{Quot}_X^{P,V}$.

For the proof, see [HL], Sec 2.2.

Naively, $\mathrm{Quot}_X^{P,m}$ can be viewed as a scheme whose points parameterize quotients of $\mathcal{O}_X^{\oplus \dim V}$ with given Hilbert polynomial.

Now we get back to our original problem: $X=C$ & we want to classify semistable rk r deg d bundles on C. Tensoring w. $\mathcal{O}(1)$ gives an equivalent classification problem but (r,d) changes to $(r, d+rd_0)$. So we can assume that $d > r(2g-1)$. Thanks to results of Sec 2.1 & Example above we see that from a semistable bundle $\mathcal{F}$ of rk r & deg d and an <u>isomorphism</u> $H^0(\mathcal{F}) \xrightarrow{\sim} V$ we get a point of $Q := \mathrm{Quot}_C^{P,V}$, where

$P(m) = (d+(1-g)r) + rd_0 \cdot m$ & $\dim V = \dim H^0(\mathcal{F}) = [H^1(\mathcal{F})=0] = \chi(\mathcal{F}) = d+(1-g)r$.

And we do get an action of $GL(V)$ on Q via $GL(V) \curvearrowright V$. A choice of identification $H^0(\mathcal{F}) \xrightarrow{\sim} V$ gives an example of an additional datum as in Sec 1.3. Note that the center of $GL(V)$ acts trivially: it preserves every quotient of $V \otimes \mathcal{O}_C$. So we get a $PGL(V)$ action on Q. For $q \in Q$ let $V \otimes \mathcal{O}_C \xrightarrow{\varphi_q} \mathcal{F}_q$ denote the corresponding data.

The next lemma describes basic properties of the action.

Lemma: 1) The following loci of $q \in Q$ are $PGL(V)$-stable:

$\mathcal{F}_q$ is a vector bundle; $H^0(\varphi_q)$ is iso; $\mathcal{F}_q$ is semi/stable vector bundle.

2) Let $q, q' \in Q$ be s.t. $H^0(\varphi_q), H^0(\varphi_{q'})$ are iso. TFAE:

(a) $q' \in PGL(N).q$

(b) $\mathcal{F}_q \simeq \mathcal{F}_{q'}$ (as coherent sheaves on C).

3) For q as in 2), we have $Stab_{GL(V)}(q) \xrightarrow{\sim} Aut(\mathcal{F}_q)$ (w. $Z(GL(V))$ mapping to scalar automorphisms).

Proof:

Note that the action of GL doesn't change the isomorphism class of $V \otimes \mathcal{O}_C \twoheadrightarrow \mathcal{F}$ (in the category of morphisms from $V \otimes \mathcal{O}_C$) hence of $\mathcal{F}$. This proves 1) & (a) $\Rightarrow$ (b) is 2).

Let's prove b) $\Rightarrow$ a). If $H^0(\varphi_q)$ is iso, then $\mathcal{F}_q$ is generated by global sections, equivalently the natural homomorphism $H^0(\mathcal{F}_q) \otimes \mathcal{O}_C \to \mathcal{F}_q$ is surjective. Let $\varphi: \mathcal{F}_q \xrightarrow{\sim} \mathcal{F}_{q'} \rightsquigarrow$ automorphism

$$V \xrightarrow[\sim]{H^0(\varphi_q)^{-1}} H^0(\mathcal{F}_q) \xrightarrow[\sim]{H^0(\varphi)} H^0(\mathcal{F}_{q'}) \xrightarrow[\sim]{H^0(\varphi_{q'})} V$$

By construction, the resulting element of $GL(V)$ sends q to q'.

To prove 3) in an exercise $\square$