Lecture 22

1) Quot schemes, cont'd
2) Projective GIT.

Ref: [Ne], Secs 1.1, 1.2, 5.1-5.3; [HL], Sec 2.2, 4.2.

1) Quot schemes, cont'd

1.0) Reminder

Let C be a smooth projective curve of genus g; $r\in\mathbb{Z}_{>0}$ & $d\in\mathbb{Z}$. We assume $d>r(2g-1)$. This is not restrictive from the point of classifying semi/stable vector bundles as we can change the degree by a multiple of r via tensoring w. a line bundle.

We have seen in Sec 2.1 of Lec 21 that every semistable rank r degree d vector bundle $\mathcal{F}$ on C has $H^1(\mathcal{F})=0$ & $H^0(\mathcal{F})\otimes\mathcal{O}_C \twoheadrightarrow \mathcal{F}$. Hence $\dim H^0(\mathcal{F})=N:=d+(1-g)r$ & the Hilbert polynomial of $\mathcal{F}$ is $P(m):=\chi(\mathcal{F}(m))=N+rd_0\cdot m$, where $\mathcal{O}(1)$ is a very ample line bundle on C & $d_0=\deg\mathcal{O}(1)$.

Fix a vector space V of dimension N. In Sec 2.2 we have stated the existence of a projective scheme $Q:=Quot_C^{P,V}$ s.t. that $Q\times C$ comes w. a quotient $\mathcal{F}^{un}$ of $V\otimes\mathcal{O}_{Q\times C}$ satisfying

- $\mathcal{F}^{un}_{\{q\}\times C}$ has Hilbert polynomial P $\forall q\in Q$ & $\mathcal{F}^{un}$ is flat/Q.
- and for every scheme S & quotient $\mathcal{F}^S$ of $V\otimes\mathcal{O}_{S\times C}$ w. analo-

gous properties $\exists!\ \varphi: S \to Q$ & unique $\mathcal{F}^C \xrightarrow{\sim} \varphi^*\mathcal{F}^{un}$.

In particular, the $\mathbb{C}$-points $q \in Q$ are in bijection w. quotients $\mathcal{F}$ of $V\otimes \mathcal{O}_C$ that have Hilbert polynomial P, equivalently, $rk(\mathcal{F})=r$, $deg(\mathcal{F})=d$. We write $\mathcal{F}_q$ for the corresponding quotient of $V\otimes\mathcal{O}_C$ & φ_q for the projection $V\otimes \mathcal{O}_C \twoheadrightarrow \mathcal{F}_q$. Note that $\mathcal{F}_q$ is nothing else but $\mathcal{F}^{un}|_{\{q\}\times C}$.

We've introduced an action of $PGL(V)$ on Q (by acting on the first tensor factor in $V\otimes\mathcal{O}_C$). If $\mathcal{F}$ is semistable, then from an identification $\iota: V \xrightarrow{\sim} H^0(\mathcal{F})$, we get a point $q(\mathcal{F},\iota)\in Q$ via $V\otimes \mathcal{O}_C \xrightarrow{\sim} H^0(\mathcal{F})\otimes\mathcal{O}_C \xrightarrow{canon.} \mathcal{F}$. Lemma in Sec 2.2 of Lec 21 implies that the $PGL(V)$-orbits of such points are in bijection w. iso. classes of semistable sheaves. So we are reduced to a problem of parameterizing orbits for a reductive group action on a scheme.

Rem: Part 3) of that lemma implies that $Stab_{PGL(V)}(q(\mathcal{F},\iota)) \xrightarrow{\sim} Aut(\mathcal{F})/\{scalars\}$. By property 4) of in Sec 1.1 of Lec 21, the right hand side is trivial if $\mathcal{F}$ is stable. So the action of $PGL(V)$ on the locus in Q corresponding to stable sheaves is free.

1.1) Open loci

Consider the following loci:

$Q^0 = \{q \in Q \mid \mathcal{F}_q$ is a vector bundle on $C\}$

$Q^1 = \{q \in Q^0 \mid H^0(\varphi_q): V \xrightarrow{\sim} H^0(\mathcal{F}_q)\}$

The main result of this section is as follows.

Lemma: $Q^1 \subset Q^0$ are Zariski open in Q.

Proof:

Let $\pi: Q\times C \longrightarrow C$ be the projection, it's proper. Let $(Q\times C)^\circ$ be the locus, where the coherent sheaf $\mathcal{F}^{un}$ is a vector bundle. A general result (based on the Nakayama lemma) tells us that $(Q\times C)^\circ$ is open in $Q\times C$, so $Q \setminus Q^\circ = \pi(Q\times C \setminus (Q\times C)^\circ)$ is closed b/c π is proper $\Rightarrow Q^\circ$ is open.

To prove Q^1 is open, we argue as follows. Let G denote the kernel of $\mathcal{O}_{Q\times C} \twoheadrightarrow \mathcal{F}^{un}$. Since $\mathcal{O}_{Q\times C}, \mathcal{F}^{un}$ are flat over Q, so is G & have SES $0 \to G_q \to V\otimes \mathcal{O}_Q \to \mathcal{F}_q \to 0$ (exercise on Tor's). Note that $\chi(\mathcal{F}_q) = N$ $\forall q\in Q$. It follows that $\dim H^0(\mathcal{F}_q) = N \Leftrightarrow \dim H^1(\mathcal{F}_q) = 0$ and both are minimal possible values. Note that $Q^1 = \{q\in Q^\circ \mid \dim H^0(\mathcal{F}_q) = N, \dim H^0(G_q) = 0\}$. This is open by the upper-semicontinuity theorem in [Ha], Ch. 3, Sec 12. $\square$

1.2) Smoothness & dimension

Lemma: Q' is smooth & every component has dimension $N^2+r^2(g-1)$.

Sketch of proof:

For $k>0$, let $A_k := \mathbb{C}[\varepsilon]/(\varepsilon^k)$, $Z_k := \operatorname{Spec}(A_k)$

We have $T_q Q' = \{\varphi: Z_2 \to Q \mid \varphi(pt)=q\} \xrightarrow{\sim} \{A_2\text{-flat quotients } \mathcal{F}^2$ of $V\otimes \mathcal{O}_C \otimes A_2$ that specialize to $\mathcal{F}_q\} \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{O}}(G_q, \mathcal{F}_q)$ (exercise: hint – think about the tangent space to Grassmanian).

Moreover, Q' is smooth at $q \Leftrightarrow$ any morphism $Z_2 \to Q'$ sending pt to q extends to Z_k, $\forall k>0$. The obstructions to extend from Z_k to Z_{k+1} live in $\operatorname{Ext}^1_{\mathcal{O}_C}(G_q, \mathcal{F}_q)$. This space fits into an exact sequence: $0 = [H^1(\mathcal{F}_q)=0 \ \forall q\in Q'] = \operatorname{Ext}^1(V\otimes\mathcal{O}_C, \mathcal{F}_q) \to \operatorname{Ext}^1(G_q, \mathcal{F}_q) \to \operatorname{Ext}^2(\mathcal{F}_q, \mathcal{F}_q)$ $[\dim C = 1] = 0$.

So $\operatorname{Ext}^1(G_q, \mathcal{F}_q) = 0$, hence q is a smooth point. We have
$\dim T_q Q' = \chi(\mathcal{H}om_{\mathcal{O}_C}(G_q, \mathcal{F}_q)) = [\text{Riemann-Roch}] = \deg \mathcal{H}om(G_q, \mathcal{F}_q)$ $+ (1-g)\operatorname{rk} \mathcal{H}om(G_q, \mathcal{F}_q) = \operatorname{rk} G_q \cdot \deg \mathcal{F}_q - \operatorname{rk} \mathcal{F}_q \cdot \deg G_q + (1-g)(N-r)r$ $= [N = d + (1-g)r] = N^2 + r^2(g-1)$. □

Since $PGL(V)$ acts freely on the locus in Q' corresponding to the stable bundles we expect the moduli space of $\dim = r^2(g-1)+1$.

Remark: Stable bundles w. given rank & degree may fail to exist. For example, Lemma implies that any stable bundle on $\mathbb{P}^1$ is a line bundle (of course, this follows from the classical fact that every vector bundle on $\mathbb{P}^1$ is the direct sum of line bundles). For $g=1$, a stable bundle exists $\Leftrightarrow GCD(r,d)=1$, the moduli space in this case is an elliptic curve. For $g>1$, a stable bundle exists $\forall\, r>0$ & d.

2) Projective GIT

Our setup is the following. Let G be a reductive group, V be a finite dimensional vector space. Suppose G acts on $\mathbb{P}(V)$ via a homomorphism $G \to PGL(V)$. Let $X \subset \mathbb{P}(V)$ be a G-stable closed subscheme, so that we get a very ample line bundle $\mathcal{O}(1)$ on X.

2.1) GIT quotients.

Let $\mathcal{H}$ denote the line bundle $\mathcal{O}(1)^{\otimes \dim V}$. The choice is motivated by the following observation. Note that by the construction the action of $SL(V)$ on $\mathbb{P}(V)$ lifts to an $SL(V)$-action on $\mathcal{O}(-1)$, in a more educated language, $\mathcal{O}(-1)$ is an $SL(V)$-equivariant line bundle. The action of $SL(V)$ on $\mathcal{O}(-1)$ doesn't factor through $PGL(V)$ (the center acts by a nontrivial character), but the action on $\mathcal{O}(-1)^{\otimes \dim V}$ does. So $\mathcal{H}$ is $PGL(V)$-equivariant.

In particular, G acts on the homogeneous coordinate ring $\tilde{A} := \bigoplus_{n\geq 0} \Gamma(X, \mathcal{H}^{\otimes n})$. Set $\tilde{X} := \operatorname{Spec} \tilde{A}$. So on $\tilde{X}$ we have an action of $\tilde{G} := G \times \mathbb{C}^{\times}$, where $\mathbb{C}^{\times}$ acts according to the grading.

Let $\theta : G \times \mathbb{C}^{\times} \to \mathbb{C}^{\times}$, $(g, z) \mapsto z$, so that $\tilde{X} /\!\!/^{\theta} \mathbb{C}^{\times} = X$.

Definition: We define the GIT quotient $X /\!\!/^{\mathcal{H}} G := \tilde{X} /\!\!/^{\theta} \tilde{G}$

For $\tilde{x} \in \tilde{X} \setminus \{0\}$, let $[\tilde{x}] := \mathbb{C}^{\times} \tilde{x} \in X$. Note that $\mathbb{C}^{\times}$ acts on $\tilde{X} \setminus \{0\}$ freely, & X is the space of orbits.

Definition: We say that

1) $[\tilde{x}]$ is $\mathcal{H}$-semistable if $\tilde{x}$ is θ-semistable
2) $[\tilde{x}]$ is $\mathcal{H}$-polystable $\Leftrightarrow$ $\tilde{x} \in \tilde{X}^{\theta\text{-ss}}$ & $\tilde{G}\tilde{x}$ is closed in $\tilde{X}^{\theta\text{-ss}}$.
3) $[\tilde{x}]$ is $\mathcal{H}$-stable $\Leftrightarrow$ $[\tilde{x}]$ is $\mathcal{H}$-polystable & $\dim G[\tilde{x}] = \dim G$.

These can be described somewhat alternatively. To a section $\sigma \in \Gamma(X, \mathcal{H}^{\otimes n})$ we assign its non-vanishing locus $X_\sigma \subset X$, an open affine subvariety. If σ is G-invariant, X_σ is G-stable

Exercise: 1) $X^{\mathcal{H}\text{-ss}} = \bigcup_{\sigma} X_\sigma$, where the union is taken over all nonzero G-invariant sections of $\mathcal{H}^{\otimes n}$ $(n > 0)$.

2) $[\tilde{x}]$ is $\mathcal{H}$-polystable $\Leftrightarrow G[\tilde{x}]$ is closed in $X^{\mathcal{H}\text{-ss}} \Leftrightarrow G[\tilde{x}]$ is closed in X_{σ} for σ as above, nonzero at $[\tilde{x}]$

3) $[\tilde{x}]$ is $\mathcal{H}$-stable $\Leftrightarrow \tilde{G}\tilde{x}$ is closed in $\tilde{X}^{\theta\text{-ss}}$ & has $\dim = \dim \tilde{G}$.

Remark: Let X be a projective scheme acted on by a reductive group G. It makes sense to speak about G-equivariant (a.k.a. G-linearized) line bundles on X (note that the equivariance is an additional structure: one can twist the action by a character of G). The construction above in this section makes sense $\forall$ G-linearized ample line bundle on X.

In a bonus section we'll discuss conditions ensuring that a line bundle is linearizable (i.e. admits a G-linearization).

2.2) Hilbert-Mumford type theorems.

We need a version of Theorem from Sec 1.1 of Lec 18 in the projective setup. Let $\gamma: \mathbb{C}^\times \to G$ be a 1-parameter subgroup, & $x \in X$. Since X is projective, $\lim_{t\to 0} \gamma(t)x$ exists, denote this point by x'. Consider the fiber $\mathcal{H}_{x'}$. Since $\mathcal{H}$ is G-equivariant, it's $\mathbb{C}^\times$-equivariant & hence $\mathcal{H}_{x'}$ is a 1-dimensional representation of $\mathbb{C}^\times$, so $\exists!$ $r \in \mathbb{Z}$ w. $\gamma(t)h = t^r h$ $\forall$ $h \in \mathcal{H}_{x'}$

$$\mu^{\mathcal{H}}(x,\gamma) := r$$

Thm: x is semistable (resp. stable) if $\mu^{\mathcal{H}}(x,\gamma)\geq 0$ (resp. $\mu^{\mathcal{H}}(x,\gamma)>0$ for each nontrivial γ).

Proof:

We can assume that $\tilde{A}$ is generated as an algebra by its deg 1 component. Let $V=\tilde{A}_1^*$ so that $\tilde{X}\hookrightarrow V$, a closed $\tilde{G}$-equivariant embedding, where $\mathbb{C}^\times \triangleleft \tilde{G}$ acts on V by $(z,v)\mapsto z^{-1}v$.

Set $V_n(\gamma)=\{v\in V \mid \gamma(t)v=t^n v\}$ & decompose $\tilde{x}$ as $\sum_n \tilde{x}_n$ w. $\tilde{x}_n\in V_n(\gamma)$. Let $n_0=\min\{n \mid \tilde{x}_n\neq 0\}$. Then $\lim_{t\to 0}\gamma(t)[\tilde{x}]=[\tilde{x}_{n_0}]$ & $\mathbb{C}\tilde{x}_{n_0}=\mathcal{H}^{-1}|_{x_0}$, so $-r=n_0$. Note that for $\tilde{\gamma}:\mathbb{C}^\times\to G\times\mathbb{C}^\times$ of the form $t\mapsto(\gamma(t),t^\ell)$ TFAE

1) $\lim_{t\to 0}\tilde{\gamma}(t)\tilde{x}$ exists

2) $\langle\theta,\tilde{\gamma}\rangle=\ell\leq n_0$ (note that $\tilde{\gamma}(t)\tilde{x}=t^{-\ell}\gamma(t)\tilde{x}$)

Let's now prove the semistability part of the theorem. By definition, x is $\mathcal{H}$-semistable $\Leftrightarrow$ $\tilde{x}$ is θ-semistable. And by Thm in Sec 1.1 of Lec 18, $\tilde{x}$ is θ-semistable $\Leftrightarrow$ $\forall\,\tilde{\gamma}:\mathbb{C}^\times\to\tilde{G}$ s.t. 1) holds, have $\langle\theta,\tilde{\gamma}\rangle\leq 0$. By equivalence of 1) & 2), we are done.

We proceed to the stability part. By Thm in Sec 1.1 of Lec 18, $\tilde{G}\tilde{x}$ is closed in $\tilde{X}^{\theta\text{-}ss}$ $\Leftrightarrow$ $\forall\,\tilde{\gamma}:\mathbb{C}^\times\to\tilde{G}$ satisfying $\langle\theta,\tilde{\gamma}\rangle=0$ & $\lim_{t\to 0}\tilde{\gamma}(t)\tilde{x}$ exists in $\tilde{X}$, we have $\lim_{t\to 0}\tilde{\gamma}(t)\tilde{x}\in G\tilde{x}$. Note that $\operatorname{im}\tilde{\gamma}$ stabilizes the limit, impossible if $\dim\tilde{G}\tilde{x}=\dim\tilde{G}$. So we arrive at the equivalence of the following two claims

a) $\tilde{G}\tilde{x}$ is closed in $\tilde{X}^{\theta\text{-}ss}$ & has $\dim = \dim G$.

b) If $\lim\limits_{t\to 0} \tilde{\gamma}(t)\tilde{x}$ exists in $\tilde{X}$, then $\langle\theta,\tilde{\gamma}\rangle < 0$.

To finish the proof of the stability part is exercise $\square$

2.2) Bonus: more on linearizability (Ref: [MF], Sec I.3)

We start with the following result

Theorem: Let X be a normal variety & $\mathcal{L}$ be a line bundle. Let G be a factorial (as a variety) connected algebraic group acting on X. Then $\mathcal{L}$ is G-linearizable.

Examples: 1) Any unipotent group is factorial.

2) Any torus is factorial.

3) Any <u>simply connected</u> semisimple group G is factorial. Indeed, the open Bruhat cell $G^0 = N^- \times T \times N \subset G$ is factorial. Any then one can show that every divisor in the complement of $G \setminus G^0$ is principal.

4) The Levi decomposition then implies that $\forall$ connected G $\exists$ factorial G' s.t. $G' \twoheadrightarrow G$ w. finite central kernel. This implies the following claim:

Corollary: Let X & $\mathcal{L}$ be as in Theorem & G be an arbitrary connected algebraic group. Then $\mathcal{L}^{\otimes n}$ is linearizable if n divides $\# \ker[G' \twoheadrightarrow G]$

Now we are going to give an application of Thm. Let G be factorial & H be an algebraic subgroup. Consider the homogeneous space G/H, it's normal. Let $\operatorname{Pic}(G/H)$ & $\operatorname{Pic}^G(G/H)$ denote the Picard groups of ordinary & linearized G-bundles. The following exercise allows to compute $\operatorname{Pic}(G/H)$.

Premium exercise: 1) Establish an isomorphism $\operatorname{Pic}^G(G/H) \xrightarrow{\sim} \mathfrak{X}(H)$.
2) Establish an exact sequence

$$\mathfrak{X}(G) \xrightarrow{res} \mathfrak{X}(H) \longrightarrow \operatorname{Pic}(G/H)$$