Lecture 24, 4/16/25

1) Comparing stabilities

Refs: [HL], Sec 4.4.

1) Comparing stabilities.

1.0) Reminder & goals.

Let C be a smooth projective curve over $\mathbb{C}$ of genus g. Our goal is to classify (semi)stable vector bundles of rk r & deg d. Via tensoring w. a suitable line bundle, we can assume d is as large as we want, in particular, $d>(2g-1)r$. For semistable $\mathcal{F}$ we have $\dim H^0(\mathcal{F})=\chi(\mathcal{F})=N:=d+(1-g)r$. Fix a vector space of this dimension, V. An identification $\iota: V \xrightarrow{\sim} H^0(\mathcal{F})$ realizes $\mathcal{F}$ as a quotient of $V\otimes\mathcal{O}_C$ giving rise to a point $q(\mathcal{F},\iota)\in Q$, the Quot scheme parameterizing the quotients $\mathcal{F}$ of $V\otimes \mathcal{O}_C$ with rank r & deg d, equivalently Hilbert polynomial $P(t)=N+rd_0t$, $d_0=\deg\mathcal{O}_C(1)$.

The group $PGL(V)$ acts on Q. In Sec 1.1 of Lec 23 we have constructed, for an integer ℓ big enough (depending on the data of the problem: g,d & r) an $SL(V)$-linearized very ample line bundle $\mathcal{H}_\ell$. We have obtained the following (Sec 1.3 of Lec 23):

I) A point $q=[V\otimes\mathcal{O}_C\twoheadrightarrow\mathcal{F}]\in Q$ is $\mathcal{H}_\ell$-(semi)stable iff $\forall$ nontrivial subspaces $V'\subsetneq V$ we have, for the image $\mathcal{F}'$ of $V'\otimes\mathcal{O}_C$,

(semi) stable

$$(1)\qquad P_{\mathcal{F}'}(\ell)\ (\geqslant)>\frac{\dim V'}{\dim V}P(\ell)$$

Here $P_{\mathcal{F}'}$ denotes the Hilbert polynomial of $\mathcal{F}'$

II) If $q=[V\otimes\mathcal{O}_C\xrightarrow{\varphi}\mathcal{F}]$ is $\mathcal{H}_\ell$-semistable, then $H_0(\varphi):V\hookrightarrow H^0(\mathcal{F})$.

Our main goal in this lecture is to prove the following theorem.

Thm: $\exists\, d(g,r)\in\mathbb{Z}_{>0}$ & for each $d\geqslant d(g,r)$ also $\ell(d,g,r)\in\mathbb{Z}_{>0}$ s.t. $\forall\, d\geqslant d(g,r)$, $\ell\geqslant\ell(d,g,r)$, $\forall q\in Q$ TFAE:

(a) q is $\mathcal{H}_\ell$-(semi)stable

(b) $q=q(\mathcal{F},\iota)$ for (semi)stable $\mathcal{F}$.

Less formally, for d & ℓ large enough, $\mathcal{H}_\ell$-(semi)stability is equivalent to the usual (semi)stability.

1.1) A version of (I) for $\ell\gg 0$

For $f_i(t)=a_i+b_it$, $a_i,b_i\in\mathbb{Q}$, $i=1,2$, we write $f_1<f_2$ iff either $b_1<b_2$ or $b_1=b_2$ but $a_1<a_2$, equivalently $f_1(\ell)<f_2(\ell)$ for $\ell\gg 0$.

Here's how this is relevant for our purposes:

Exercise 1: For $\mathcal{F}_2 \in Coh(C)$, $\mathcal{F}_1 \subsetneq \mathcal{F}_2 \Rightarrow P_{\mathcal{F}_1} < P_{\mathcal{F}_2}$.

Let $q = [V \otimes \mathcal{O}_C \xrightarrow{\varphi} \mathcal{F}] \in Q$ & $\mathcal{F}' \subset \mathcal{F}$. We write $V_{\mathcal{F}'}$ for the pre-image of $H^0(\mathcal{F}') \subset H^0(\mathcal{F})$ in V under $H^0(\varphi): V \to H^0(\mathcal{F})$.

Proposition: $\exists\ \ell(d,g,r) \in \mathbb{Z}_{>0}$ s.t. $\forall\ \ell \geq \ell(d,g,r)$, $q \in Q$ is H_ℓ-(semi) stable iff

(2) $\qquad \dim V \cdot P_{\mathcal{F}'}(\geq) > \dim V_{\mathcal{F}'} \cdot P_{\mathcal{F}}$ $\forall$ nonzero $\mathcal{F}' \subsetneq \mathcal{F}$.

In the proof we will need:

Exercise 2 $\{\deg(\mathcal{F}') \mid \mathcal{F}' \subset \mathcal{F}\}$ is bounded from above. Hint: induction on $rk(\mathcal{F})$ & note that every vector bundle contains a rk 1 subbundle.

Proof of Proposition: Note that $\{(\deg \mathcal{F}', rk\, \mathcal{F}') \mid \mathcal{O}_C^{\oplus N} \twoheadrightarrow \mathcal{F}' \hookrightarrow \mathcal{F}\}$ is finite: indeed observe that $rk\, \mathcal{F}' \in \{0, \dots N\}$ & $\deg(\mathcal{F}') \geq 0$ b/c $\mathcal{O}_C^{\oplus N}$ is semistable, then use Exercise. So $\{P_{\mathcal{F}'} : \mathcal{O}_C^{\oplus N} \twoheadrightarrow \mathcal{F}' \hookrightarrow \mathcal{F}\}$ is finite. Recall that $f_1 < f_2 \Leftrightarrow f_1(\ell) < f_2(\ell)$ for ℓ large enough (depending on f_1, f_2). So, for $\ell \gg 0$ we can replace 1) in I) by

(1') $P_{\mathcal{F}'} \geq \frac{\dim V'}{\dim V} P_{\mathcal{F}}$ $\forall$ images $\mathcal{F}'$ of $V' \otimes \mathcal{O}_C \to \mathcal{F}$.

Now notice that for such $\mathcal{F}'$ we have $V' \subset V_{\mathcal{F}'}$. So if (2) holds, then

$P_{\mathcal{F}'}(\geq) > \frac{\dim V_{\mathcal{F}'}}{\dim V} P \geq [P \text{ has positive coefficients}] \frac{\dim V'}{\dim V} P$ implying (1'). Conversely, let $\underline{\mathcal{F}}'$ be the image of $V_{\mathcal{F}'} \otimes \mathcal{O}_C$ in $\mathcal{F}$, then $\underline{\mathcal{F}}' \subset \mathcal{F}' \Rightarrow$ [Exercise 1] $P_{\underline{\mathcal{F}}'} \leq P_{\mathcal{F}'}$, so (1') implies (2). $\square$

1.2) Criterion of (semi)stability.

Some more notation: for $M \in Coh(C)$ we write $h^i(M)$ for $\dim H^i(M)$, so that $\chi(M) = h^0(M) - h^1(M)$.

Proposition: Let $\mathcal{F} \in Coh(C)$ have rk r & deg d. Then $\exists\, d(g,r)$ s.t. $\forall\, d \geq d(g,r)$ TFAE

(a) $\mathcal{F}$ is (semi)stable

(b) $\forall\, r' \in [0,r]$ & all subsheaves $\{0\} \subsetneq \mathcal{F}' \subsetneq \mathcal{F}$ of rk r' we have $h^0(\mathcal{F}') (\leq) < \frac{r'}{r} N$

(c) $\forall\, r'' \in [0,r]$ & all quotient sheaves $\mathcal{F}''$ of $\mathcal{F}$ different from 0 & $\mathcal{F}$ we have $h^0(\mathcal{F}'') (\geq) > \frac{r''}{r} N$

Moreover, if $\mathcal{F}$ is semistable, then $=$ in b) (resp. c)) implies $\mu(\mathcal{F}') = \mu(\mathcal{F})$ (resp., $\mu(\mathcal{F}'') = \mu(\mathcal{F})$).

Ideas of proof: (b) $\Rightarrow$ (a) $\Rightarrow$ (c) follow from two easy observations:

(1) $h^0(M) \geq \chi(M)\ \forall\, M \in Coh(C)$

(2) $\mathcal{F}$ is (semi)stable $\Leftrightarrow \chi(\mathcal{F}') (\leq) < \frac{r'}{r} \chi(\mathcal{F}) \Leftrightarrow \chi(\mathcal{F}'') (\geq) > \frac{r''}{r} \chi(\mathcal{F})$

The other implications are hard: they require a boundedness argument (cf. Fact in Sec 1.1.2 of Lec 23) and a bound on $h^0(\mathcal{F}')$ for $\mathcal{F}'\subset\mathcal{F}$ w. small slope. See [HL], Thm 4.4.1 for details. $\square$

1.3) Proof of Thm

We take $d(g,r)$ as in Sec 1.2 & assume $d\geqslant d(g,r)$. Then take $\ell(d,g,r)$ as in Sec 1.1. For $\ell\geqslant\ell(d,g,r)$, the $\mathcal{H}_\ell$-(semi) stability is controlled by Proposition in Sec 1.1. (Semi) stability of $\mathcal{F}$ will be studied using Proposition in Sec 1.2.

Step 1: We show (a)$\Leftrightarrow$(b) assuming $q\in Q^1=\{[V\otimes\mathcal{O}_C\xrightarrow{\psi}\mathcal{F}\mid H^0(\psi)$ is iso & $\mathcal{F}$ is vector bundle$\}$. Note that here $V_{\mathcal{F}'}=H^0(\mathcal{F}')\Rightarrow \dim V_{\mathcal{F}'}=h^0(\mathcal{F}')$. So (b)$\Leftrightarrow h^0(\mathcal{F}')r\ (\leqslant)< r'N$; (a)$\Leftrightarrow h^0(\mathcal{F}')P\ (\leqslant)< NP_{\mathcal{F}'}$. The leading coefficients of $h^0(\mathcal{F}')P$ & $NP_{\mathcal{F}'}$ are $h^0(\mathcal{F}')rd_0$ & $Nr'd_0$. So (a)$\Rightarrow$(b); and if $h^0(\mathcal{F}')r<r'N$, then $h^0(\mathcal{F}')P<NP_{\mathcal{F}'}$, yielding the stable case of (b)$\Rightarrow$(a). For the semistable part, it remains to consider the case $h^0(\mathcal{F}')r=r'N$, which by Proposition in Sec 1.2 means $\mu(\mathcal{F}')=\mu(\mathcal{F})$. So $\mathcal{F}'$ is semistable $\Rightarrow$ $h^0(\mathcal{F}')=\chi(\mathcal{F}')$ & $\chi(\mathcal{F}')/\chi(\mathcal{F})=P_{\mathcal{F}'}/P=r'/r\Rightarrow h^0(\mathcal{F}')P=NP_{\mathcal{F}'}$.

In particular, (b)$\Rightarrow q\in Q^1$ as we've seen already in Lec 21. So (b)$\Rightarrow$(a).

Step 2: To show (6) $\Rightarrow$ (a) it remains to show $Q^{H_e\text{-ss}} \subset Q^1$.
Assume $q = [V \otimes \mathcal{O}_C \xrightarrow{\psi} \mathcal{F}] \in Q^{H_e\text{-ss}} \setminus Q^1$. Let $\underline{\mathcal{F}} := \mathcal{F}/\mathrm{tor}(\mathcal{F})$. We can find a subsheaf $\mathcal{E} \subset \mathcal{F} \otimes m_x^{\otimes -k}$ ($x \in C$, $k \gg 0$) containing $\underline{\mathcal{F}}$ & of deg d (& rk r). Our first goal is to show $\mathcal{E}$ is semistable.

Let $\mathcal{E}''$ be a quotient of $\mathcal{E}$ of rank r''. Let $\mathcal{F}'$ be the kernel of $\mathcal{F} \to \mathcal{E} \to \mathcal{E}''$. Passing to leading coefficients in inequality $\dim(V) \cdot P_{\mathcal{F}'} \geq \dim(V')P$ (and dividing by d_0) we get

(3) $$N \cdot r' \geq \dim V_{\mathcal{F}'} \cdot r$$

Now we have the following inequalities:

$h^0(\mathcal{E}'') \geq [0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{E}''] \; h^0(\mathcal{F}) - h^0(\mathcal{F}') \geq [V/V_{\mathcal{F}'} \hookrightarrow H^0(\mathcal{F})/H^0(\mathcal{F}')]$ $N - \dim V_{\mathcal{F}'} \geq [(3)] \; N - \frac{r'}{r}N = \frac{r''}{r}N$. Then applying Proposition in Sec. 1.2 we see that $\mathcal{E}$ is s/stable (rk r & deg d). It follows that

(4) $h^0(\mathcal{E}) = N$.

Let ι denote the composition $\mathcal{F} \twoheadrightarrow \underline{\mathcal{F}} \hookrightarrow \mathcal{E}$ so that we have $\ker \iota = \mathrm{tor}(\mathcal{F})$. By II), $H_0(\psi): V \hookrightarrow H^0(\mathcal{F})$. Furthermore, we claim $\mathrm{im}\, H_0(\psi) \cap \mathrm{tor}(\mathcal{F}) = \{0\}$. Otherwise, take $\mathcal{F}'$ for this intersection $\Rightarrow \dim V_{\mathcal{F}'} \neq 0$ & $r' = 0$, contradicting (3). It follows that $H^0(\iota \circ \psi): V \hookrightarrow H^0(\mathcal{E})$. Combining this with (4), we see that $H^0(\iota) \circ H^0(\psi) = H^0(\iota \circ \psi)$ is iso.

We have the following commutative diagram:

$$\begin{tikzcd}
V\otimes\mathcal{O}_C \arrow[r, hook] \arrow[rr, bend left, "\sim"] \arrow[dr] & H^0(\mathcal{F})\otimes\mathcal{O}_C \arrow[r] \arrow[d] & H^0(\mathcal{E})\otimes\mathcal{O}_C \arrow[d, two heads] \\
& \mathcal{F} \arrow[r] & \mathcal{E}
\end{tikzcd}$$

$\leftarrow$ $\mathcal{E}$ is s/stable

So the composition $V\otimes\mathcal{O}_C \to \mathcal{F} \to \mathcal{E}$ is an epimorphism, hence $\mathcal{F} \twoheadrightarrow \mathcal{E}$. Since $\mathrm{rk}\,\mathcal{F} = \mathrm{rk}\,\mathcal{E}$ & $\deg\mathcal{F} = \deg\mathcal{E} \Rightarrow \mathcal{F} \xrightarrow{\sim} \mathcal{E}$ & $q = q(\mathcal{E}, H^0(\iota\circ\varphi)) \Rightarrow q \in Q^1$. $\square$

1.4) Bonus – generalization: (semi) stable sheaves

The construction of moduli spaces of stable vector bundles on smooth (projective) curves can be generalized to higher dimensions with a similar (but more technically involved) approach (explained in [HL]).

We explain the setting. Let X be a projective scheme w. a very ample line bundle $\mathcal{O}(1)$ that allows us to define the Hilbert polynomial $P_{\mathcal{F}}(t)$ of a sheaf $\mathcal{F}$. Its degree is $\dim \mathrm{Supp}(\mathcal{F})$. We consider the reduced Hilbert polynomial $p_{\mathcal{F}}(t)$: the monic polynomial proportional to $P_{\mathcal{F}}(t)$.

We call $\mathcal{F}$ pure if $\mathcal{F}$ has no nonzero subs w. support of $\dim < \dim\mathrm{Supp}(\mathcal{F})$. A pure sheaf $\mathcal{F}$ is called (semi)stable if $p_{\mathcal{F}'} (\leq) < p_{\mathcal{F}}$ $\forall\, 0 \subsetneq \mathcal{F}' \subsetneq \mathcal{F}$, where the order is similar to one defined in Sec 1.1: $p_1 < p_2$ iff $p_1(t) < p_2(t)$ for $t \ll 0$.