Lecture 25, 4/21/25

1) Moduli spaces of vector bundles: conclusion
2) Invariants for non-reductive groups

Refs: [Ne], Sec 5.3.

1) Moduli spaces of vector bundles: conclusion

In the previous lecture we have seen that the two notions of (semi)stability coincide: for vector bundles of rk r & deg $d\geqslant 0$ & for points of the Quot scheme $Q=Quot_C^{P,V}$, where we consider the GIT stability for the action of $PGL(V)$ & the linearized line bundle $\mathcal{L}=\mathcal{H}_\ell^{\otimes N}$ (w. $\ell\gg 0$). So we can construct moduli spaces of (semi)stable bundles as GIT quotients.

Denote the GIT quotient $Q/\!/^{\mathcal{L}}PGL(V)$ by $\mathcal{M}^{ss}(r,d)$. By what was explained in Sec 2.1 of Lec 22, the points of $\mathcal{M}^{ss}(r,d)$ are in bijection w. the closed orbits in $Q^{\mathcal{L}-ss}$.

Exercise 1: The orbit of $q=[V\otimes O_C \twoheadrightarrow \mathcal{F}]$ is closed iff $\mathcal{F}$ is polystable (hint: use the description of limits in Sec 1.2 of Lec 23 & Corollary in Sec 1.3 of Lec 23).

So $\mathcal{M}^{ss}(r,d)$ parameterizes iso classes of polystable bundles. Let $\mathcal{M}^s(r,d)$ denote the locus in $\mathcal{M}^{ss}(r,d)$ corresponding to the stable bundles.

Proposition: i) $\mathcal{M}^s(r,d)$ is Zariski open in $\mathcal{M}^{ss}(r,d)$.

ii) Moreover, $Q^{\mathcal{L}-s}$ (the locus of stable points) is a principal $PGL(V)$-bundle over $\mathcal{M}^s(r,d)$.

Sketch of proof:

i): follows from

Exercise: Let a reductive group G act on an affine variety X. Consider $Y=\{x\in X \mid \dim \operatorname{Stab}_G(x)>0\}$. Then Y is a closed G-stable subset & $X \setminus \pi^{-1}(\pi(Y))$ is the locus of closed G-orbits of dim $=\dim G$ in X; here $\pi: X\longrightarrow X/\!/G$ is the quotient morphism.

ii): Recall that if $\mathcal{F}$ is stable, then $\operatorname{Stab}_{PGL(V)} q(\mathcal{F},\iota)$ is trivial. Now i) follows Corollary of the Luna slice theorem in Sec 1.4 of Lec 14. $\square$

Using Sec 1.2 of Lec 22 one shows that $\mathcal{M}^s(r,d)$ is smooth of dimension $r^2(g-1)+1$. Moreover, one can show that $\mathcal{M}^s(r,d)$ is a coarse moduli space of stable bundles, see e.g. Theorem 5.8 in [Ne] and references in the proof.

2) Invariants for non-reductive groups

2.0) Setup

The base field is $\mathbb{C}$.

Throughout this class our central topic was invariants/quotients for actions of reductive groups. For non-reductive groups not much can be said in general: invariants may fail to be finitely generated (see Bonus remark in Sec 2.3 of Lec 4). So, we are going to consider a more restrictive situation.

Let G be a reductive group & H be its algebraic subgroup. We want to consider actions of H restricted from G.

In Lec 4, Sec 1, we introduced the homogeneous space G/H. It's a variety equipped w. a G-action s.t. $\mathrm{Stab}_G(1H)=H$, it's unique w. this property. Its connection to the setup above is as follows.

Proposition 1: Let A be a commutative algebra equipped w. a rational representation of G by automorphisms. Then we have an algebra isomorphism:

$$(\mathbb{C}[G/H]\otimes A)^G \xrightarrow{\sim} A^H,\ \sum f_i\otimes a_i \mapsto \sum f_i(1H) a_i$$

From this & Hilbert's thm (Prop 1 in Sec 1.0 of Lec 3) we get

Corollary: If $\mathbb{C}[G/H]$ is finitely generated, then A^H is.

2.1) Structure of $\mathbb{C}[G/H]$

We will prove the proposition after some preparation. Consider the G-equivariant projection $p: G \to G/H$, $g \mapsto gH$. The $p^*: \mathbb{C}[G/H] \hookrightarrow \mathbb{C}[G]$ is G-equivariant, in particular $\mathbb{C}[G/H]$ is a rational G-representation.

Lemma: Let V be a finite dimensional rational G-representation. Then $\mathrm{Hom}_G(V, \mathbb{C}[G/H]) \xrightarrow{\sim} (V^*)^H$.

Proof:

Note that $\mathrm{Hom}_G(V, \mathbb{C}[G/H]) \xrightarrow{\sim} \mathrm{Hom}_{G\text{-}Alg}(S(V), \mathbb{C}[G/H]) \xrightarrow{\sim}$ $[S(V) = \mathbb{C}[V^*]]$ $\mathrm{Hom}_{G\text{-}Alg}(\mathbb{C}[V^*], \mathbb{C}[G/H]) \xrightarrow{\sim} \mathrm{Mor}_G(G/H, V^*)$.
Here the last term is a set of G-equivariant morphisms $G/H \to V^*$ – which is naturally a vector space b/c V^* is. The last isomorphism is a consequence of the following general observation: to give a morphism $Y \to X$, where X is affine is the same as to give an algebra homomorphism $\mathbb{C}[X] \to \mathbb{C}[Y]$. Consider the map $\mathrm{Mor}_G(G/H, V^*) \xrightarrow{\varepsilon_{1H}} V^*$ of evaluation at $1H$, the image lies in $(V^*)^H$. ε_{1H} is clearly injective & it remains to show the image is $(V^*)^H$. Let $\alpha \in (V^*)^H$. Then $\mathrm{Stab}_G((1H, \alpha)) = H$ (for $G \curvearrowright G/H \times V^*$). So

we get an isomorphism $G/H \xrightarrow{\sim} G(1H,\alpha)$ mapping $1H$ to $(1H,\alpha)$ & hence a morphism $G/H \xrightarrow{\sim} G(1H,\alpha) \hookrightarrow G/H \times V^* \twoheadrightarrow V^*$ sending $1H$ to α $\square$

Important exercise 1: Track the construction to show that the isomorphism is given by sending $\varphi \in \mathrm{Hom}(V, \mathbb{C}[G/H])$ to $ev_{1H} \circ \varphi$, where $ev_{1H}: \mathbb{C}[G/H] \longrightarrow \mathbb{C}$, $f \mapsto f(1H)$.

Corollary: As a G-representation, $\mathbb{C}[G/H] \xrightarrow{\sim} \bigoplus_V V \otimes (V^*)^H$, where the sum is over the isomorphism classes of irreducible G-modules.

Example: Let H be a connected reductive group embedded diagonally into $G = H \times H$. The irreducible G-representations are of the form $V_1 \otimes V_2$, where $V_i \in \mathrm{Irr}(H)$. We have $(V_1 \otimes V_2)^{*,H} = \mathrm{Hom}_H(V_1^*, V_2)$, which is 1-dimensional if $V_2 \simeq V_1^*$ and is 0 else. So

$$\mathbb{C}[H] \simeq_{H \times H} \bigoplus_{V \in \mathrm{Irr}(H)} V \otimes V^*$$

Exercise 2: $p^*: \mathbb{C}[G/H] \hookrightarrow \mathbb{C}[G]$ identifies $\mathbb{C}[G/H]$ with $\mathbb{C}[G]^H$.

Note that there are other ways to establish this identification, e.g. by using that $G \to G/H$ is a principal H-bundle.

2.2) Proof of Proposition 1

Recall (cf. Exercise 1 in Sec 1.1) the map $ev_{1H}: \mathbb{C}[G/H] \to \mathbb{C}$, $f \mapsto f(1H)$, it's an algebra homomorphism & it's H-equivariant. It induces an algebra homomorphism $ev_{1H} \otimes id: \mathbb{C}[G/H] \otimes A \to A$ mapping $(\mathbb{C}[G/H]\otimes A)^G$ to A^H. We claim this restriction is an isomorphism. Indeed, A is a union of finite dimensional rational representations, V. So it's enough to show that

$$ev_{1H} \otimes id: (\mathbb{C}[G/H]\otimes V)^G \xrightarrow{\sim} V^H.$$

This follows from Lemma combined with Exercise 1 in Sec 1.1.

2.3) Case $H=U$

Starting from now we will concentrate on the special case of $H=U$, where U is a maximal unipotent subgroup of G (= the unipotent radical of a Borel, B). Let T be a maximal torus in B, so that $B = T \ltimes U$.

G/U carries an action of $\tilde{G} = G \times T$ via $(g,t).g'U = gg't^{-1}U$, so that $\mathrm{Stab}_{\tilde{G}}(1U) = \tilde{U} := \{(tu, t) \mid t \in T, u \in U\}$.

Lemma: As $G\times T$-module, $\mathbb{C}[G/U] = \bigoplus_{\lambda} V(\lambda) \otimes \mathbb{C}_{\lambda^*}$, where the sum is taken over the dominant elements of $\mathfrak{X}(T)$ & $V(\lambda)$ denote the irreducible module w. highest weight λ and λ^* denotes the highest weight of $V(\lambda)^*$.

Proof:

Proposition 1 implies

$\mathbb{C}[G/U] = \mathbb{C}[\tilde{G}/\tilde{U}] = \bigoplus_{\lambda,\mu} (V(\lambda)\otimes\mathbb{C}_\mu)\otimes[(V(\lambda)\otimes\mathbb{C}_\mu)^*]^{\tilde{U}}$. Here the summation is taken over dominant $\lambda\in\mathfrak{X}(T)$ & all $\mu\in\mathfrak{X}(T)$. Note that $U\subset\tilde{U}$ acts trivially on $\mathbb{C}_\mu^*$ & $[V(\lambda)^*]^U = V(\lambda^*)^U = \mathbb{C}_{\lambda^*}$ (the highest weight subspace) as a module over $T\subset\tilde{U}$.

We have

$$[(V(\lambda)\otimes\mathbb{C}_\mu)^*]^{\tilde{U}} = [V(\lambda^*)^U\otimes\mathbb{C}_{-\mu}]^T = \mathbb{C}^T_{\lambda^*-\mu} = \begin{cases}\mathbb{C}, & \mu=\lambda^*\\ 0, & \text{else}\end{cases}$$

This implies the claim. $\square$

Example: Suppose that $G=SL_2$. Consider the G-action on $\mathbb{C}^2$, the space of column vectors. The stabilizer of $e_1=\begin{pmatrix}1\\0\end{pmatrix}$ is U and so $G/U\hookrightarrow\mathbb{C}^2$. The image is $\mathbb{C}^2\setminus\{0\}$. It follows that $\mathbb{C}[G/U]=\mathbb{C}[x,y]$. Every irreducible representation of SL_2 occurs in $\mathbb{C}[x,y]$ with multiplicity 1. The action of $\mathbb{C}^\times=T$ commutes w. SL_2 and we have $t.e_1=t^{-1}e_1$. Hence it is given by $t.v=t^{-1}v$, $v\in\mathbb{C}^2$. It follows that on the graded component $\mathbb{C}[x,y]_n=V(n)$ it is by $t\mapsto t^n$ confirming the conclusion of the lemma.

In the next lecture we will see that $\mathbb{C}[G/U]$ is finitely generated for all G.

Corollary in Sec 1.0 & Example give the following classical result.

Theorem (Weitzenböck, 1932) The algebra of invariants of any <u>linear</u> action of the additive group $\mathbb{G}_a$ is finitely generated.

Proof:

Let $G = SL_2$ so that $U \simeq \mathbb{G}_a$. It's enough to show that any homomorphism $U \to GL(V)$ extends to $G \to GL(V)$, then we have $\mathbb{C}[V]^{\mathbb{G}_a} \xrightarrow{\sim} (\mathbb{C}[G/U] \otimes \mathbb{C}[V])^{SL_2} = \mathbb{C}[\mathbb{C}^2 \oplus V]^{SL_2}$. To give a homomorphism of algebraic groups $\varphi: \mathbb{G}_a \to GL(V)$ amounts to specifying a nilpotent element of $\mathfrak{gl}(V)$ $(= d_1\varphi(1))$. Every such element can be included into an $\mathfrak{sl}_2$-triple - this follows from the Jacobson-Morozov theorem from Sec. 2.3.1 of Lec 10 (or from the JNF theorem combined w. the classification of $\mathfrak{sl}_2$-representations). The $\mathfrak{sl}_2$-triple gives rise to an extension of φ to SL_2 finishing the proof. $\square$