INVARIANT HOMEWORK 3, DUE MAR 28

For simplicity, the base field below is $\mathbb{C}$.

Problem 1

Let m, n, d be positive integers, $G = \mathrm{GL}_d, V = \mathrm{Mat}_{n,d} \times \mathrm{Mat}_{d,m}$ with action of GL_d given by $g.(A, B) = (Ag^{-1}, gB)$. Show that for $(A, B) \in V$ the following conditions are equivalent:

(1) $G.(A, B)$ is closed,
(2) and $\ker A \oplus \operatorname{im} B = \mathbb{C}^d$.

Problem 2

Let X_d denote the variety of all matrices in $\mathrm{Mat}_{n,m}$ of rank $\leqslant d$. It is known that X_d is normal[1]. Let G, V be as in the previous problem. Prove that there is an isomorphism of X_d with $V//G$ such that the quotient morphism π becomes $(A, B) \mapsto AB$.

Problem 3

Let $F \in S^3(\mathbb{C}^3)$ be a nonzero element, and let $C := \{[a : b : c] | F(a, b, c) = 0\}$ be the corresponding degree 3 curve in $\mathbb{P}^2$. Show that $\mathrm{SL}_3 .F$ is closed if and only if one of the following conditions hold:

- C is smooth,
- C is the union of three lines not passing through the same point.

[1]we may prove this in the course invariant-theoretically, time permitting