INVARIANT HOMEWORK 5 (THE LAST), DUE MAY 5

Problem 1, 10pts

Let V be a finite dimensional vector space equipped with a rational representation of a torus T. Pick a nontrivial character θ of T. Recall that to a vector $v \in V$ we can assign a convex polytope $\mathsf{Conv}(v)$ (see Section 1.1.1 of Lecture 11). Show that

a) $v \in V^{\theta-ss}$ iff the ray connecting 0 and $-\theta$ intersects $\mathsf{Conv}(v)$.

b) And Tv is closed in $V^{\theta-ss}$ iff there is a point in the relative interior of $\mathsf{Conv}(v)$ in that intersection.

Problem 2, 10pts

Let G be a connected reductive group and X be an affine variety acted on by G. This problem analyzes the dependence of $X^{\theta-ss}$ on θ. Consider the $\mathbb{R}$-vector space $\mathfrak{X}_{\mathbb{R}} := \mathfrak{X}(G) \otimes_{\mathbb{Z}} \mathbb{R}$. Note that it has a rational structure, so it makes sense to speak about rational (linear) hyperplanes. A collection of such, Γ, partitions $\mathfrak{X}_{\mathbb{R}}$ into the disjoint union of rational cones that are open in their closures. We refer to these cones as facets of the hyperplane arrangement Γ. For example, for any nontrivial collection of hypeplanes in the 1-dimensional space we have exactly three facets.

Show that there is a finite collection of rational linear hyperplanes such that

- $X^{\theta_1-ss} = X^{\theta_2-ss}$ if θ_1, θ_2 are in the same facet.
- $X^{\theta_1-ss} \subset X^{\theta_2-ss}$ if the facet of θ_1 contains θ_2 in its closure.

Hint: you may try to reduce to the case of a linear action of a torus.