Lecture 26, 4/23/05.

1) U-invariants

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1.0) Recap

Let G be a connected reductive group over $\mathbb{C}$, $U \subset G$ be a maximal unipotent subgroup of G & $T \subset G$ be a maximal torus normalizing U. Consider the homogeneous space G/U (sometimes called the "principal affine space" even though it's not affine) It's acted on by $G\times T$ & by Sec 2.3 in Lec 25

(1) $\quad \mathbb{C}[G/U] \xrightarrow[G\times T]{\sim} \bigoplus_{\lambda\in\mathfrak{X}^+} V(\lambda)\otimes\mathbb{C}_{\lambda^*}$,

where we write $\mathfrak{X}^+$ for the monoid of dominant weights in $\mathfrak{X}(T)$.

We further know (Proposition 1 in Sec 2.0 of Lec 25) that for every commutative algebra A equipped with a rational representation of G by automorphisms we have an algebra iso

$$A^U \xleftarrow{\sim} (\mathbb{C}[G/U]\otimes A)^G$$

In particular if $\mathbb{C}[G/U]$ is finitely generated (we will see below that this is the case), then A^U is also finitely generated.

Rem: (1) equips $\mathbb{C}[G/U]$ w. a $\mathfrak{X}^+$-grading. It comes from the T-action and hence is a G-stable algebra grading. So A^U comes equipped w. an algebra grading by $\mathfrak{X}^+$: a highest vector in $V(\lambda)$ $\hookrightarrow A$ lives in degree λ^*.

1.1) Finite generation.

Theorem: For A as above TFAE:

(a) A is finitely generated

(b) A^U is finitely generated.

The scheme of the proof is as follows. We first prove (b)$\Rightarrow$(a). Then we prove the following claim of independent interest:

Proposition: $\mathbb{C}[G/U]^U \xrightarrow{\sim} \mathbb{C}\mathfrak{X}^+$, the monoid algebra of $\mathfrak{X}^+$.

Using this proposition & (b)$\Rightarrow$(a) we prove (a)$\Rightarrow$(b).

Proof of (b)$\Rightarrow$(a): Let $f_1, \dots f_k \in A^U$ be generators. Any finite collection of elements of A incl. $f_1, \dots, f_k$ is contained in a finite dimensional G-subrepresentation, say V. Let $A' \subset A$ be the subalgebra generated by V (hence finitely generated). It's G-stable

& contains A^U, i.e. all highest vectors in A. Since A is a completely reducible G-module, we have $A'=A$ $\square$

Proof of Proposition: First, consider the case when G is semisimple & simply connected. Here $\mathfrak{X}^+$ is a free monoid (generated by fundamental weights, $\omega_1,\dots\omega_r$). Recall that $\mathbb{C}[G/U]$ is graded by $\mathfrak{X}^+$ via the T-action and the grading is preserved by G. So $\mathbb{C}[G/U]^U\subset\mathbb{C}[G/U]$ is a graded subalgebra. (1) $\Rightarrow$ $\mathbb{C}[G/U]^U_\lambda=\mathbb{C}$ if $\lambda\in\mathfrak{X}^+$ & $\{0\}$ else. Note that

$$\mathbb{C}[G/U]^U\subset\mathbb{C}[G/U]\subset\mathbb{C}[G]$$

are domains (G is connected hence irreducible). Pick $f_i\in\mathbb{C}[G/U]^U_{\omega_i}\setminus\{0\}$. We claim that $\mathbb{C}[G/U]^U=\mathbb{C}[f_1,\dots f_r]$. Indeed for $d_i\in\mathbb{Z}_{\geqslant 0}$ we have $\mathbb{C}\prod_{i=1}^r f_i^{d_i}=\mathbb{C}[G/U]^U_\lambda$ w. $\lambda=\sum d_i\omega_i$. All these monomials are linearly independent (as they are in different degrees) yielding the claim. To prove $\mathbb{C}[G/U]^U\xrightarrow{\sim}\mathbb{C}\mathfrak{X}^+$ in this case note that $\mathbb{C}\mathfrak{X}^+\xrightarrow{\sim}\mathbb{C}[f_1,\dots f_r]$ via $\omega_i\mapsto f_i$.

Now consider the general case. We can present G as $(Z\times G')/\Gamma$, where Z is a torus, G' is semisimple & simply connected & Γ is finite & central. Then $U\hookrightarrow G'$ as max. unip. subgroup & $G/U\xrightarrow{\sim}(Z\times G'/U)/\Gamma\Rightarrow\mathbb{C}[G/U]^U\xrightarrow{\sim}[(\mathbb{C}[Z]\otimes\mathbb{C}[G'/U])^\Gamma]^U\xrightarrow{\sim}$ $(\mathbb{C}[Z]\otimes\mathbb{C}[G'/U]^U)^\Gamma\xrightarrow{\sim}(\mathbb{C}\mathfrak{X}(Z)\otimes\mathbb{C}\mathfrak{X}'_+)^\Gamma\xrightarrow{\sim}$ [exercise] $\mathbb{C}\mathfrak{X}^+$ $\square$

dominant wts for G'

Proof of (a) $\Rightarrow$ (b). $\mathfrak{X}^+$ is a finitely generated monoid, so $\mathbb{C}\mathfrak{X}^+$ ($\simeq \mathbb{C}[G/U]^U$) is a finitely generated algebra. Applying (b) $\Rightarrow$ (a) to $\mathbb{C}[G/U]$, we see that $\mathbb{C}[G/U]$ is finitely generated. It follows that $A^U \xrightarrow{\sim} (\mathbb{C}[G/U] \otimes A)^G$ is finitely generated $\square$

1.2) Preservation of normality

Theorem: Let A be a finitely generated domain equipped w. a rational representation of G by algebra automorphisms. TFAE:

(i) A is normal

(ii) A^U is normal.

Proof:

(i) $\Rightarrow$ (ii) is a general phenomenon that works for any automorphism group, Lemma 1 in Sec 1 of Lec 4. The proof of (ii) $\Rightarrow$ (i) is in several steps.

Step 1: Let $\tilde{A}$ denote the integral closure of A in $\operatorname{Frac}(A)$. It's finitely generated & G-stable (in $\operatorname{Frac}(A)$). We claim that the action $G \curvearrowright \tilde{A}$ is rational.

Let $X = \operatorname{Spec}(A)$, $\tilde{X} = \operatorname{Spec}(\tilde{A})$. Then the natural (dominant) morphism $\pi: \tilde{X} \to X$ has the following universal property: $\forall$ dominant morphism $\varphi: Y \to X$ from a normal (affine) variety Y uniquely factors through $\tilde{X}$.

Note that the product of normal varieties is normal, see Stacks Project, Lemma 33.10.5. So take $Y = G\times\tilde{X}$ & the composition $G\times\tilde{X} \xrightarrow{id\times\pi} G\times X \xrightarrow{a} X$, where a is the action morphism. It factors as $G\times\tilde{X} \xrightarrow{\tilde{a}} \tilde{X} \xrightarrow{\pi} X$. The following exercise implies the claim of this step, thx to Proposition in Sec 1.1 of Lec 3

Exercise: $\tilde{a}: G\times\tilde{X}\longrightarrow\tilde{X}$ is an action morphism.

Step 2: Suppose we know $\tilde{A}^U = A^U$. Since $A\subset\tilde{A}$ is G-stable & contains $\tilde{A}^U$, we get $A=\tilde{A}$ yielding (i). Since A^U is normal, it remains to show:

(a) $\tilde{A}^U\subset \mathrm{Frac}(A^U)$

(b) $\tilde{A}^U$ is integral over A^U.

Step 3: We prove (a). Let $f\in\tilde{A}^U\subset \mathrm{Frac}(A)^U$. Our job is to find $f'\in A^U$ with $ff'\in A$. Let $I=\{f_1\in A \mid f_1 f\in A\}$. This is a nonzero ideal & it's U-stable. The representation of U is rational & so on every finite dimensional subrepresentation U acts by unipotent operators. By an algebraic group version of Engel's theorem, $\exists$ nonzero U-fixed vector $f'\in I$. Then $f'f\in A$, autom. U-invariant, proving (a).

Step 4: Let $B:=\mathbb{C}[G/U]\otimes A$, $\tilde{B}:=\mathbb{C}[G/U]\otimes\tilde{A}$. Then $\tilde{B}$ is integral over B. We reduce (6) to the following:

Claim:

Let $B, \tilde{B}$ be algebras equipped w. rational G-representations by automorphisms. Let $B \to \tilde{B}$ be a G-equivariant algebra homomorphism. If $\tilde{B}$ is integral over B, then $\tilde{B}^G$ is integral over B^G.

Proof of Claim: Let $f \in \tilde{B}^G$ & $f^n + b_1 f^{n-1} + \ldots + b_0 = 0$ w. $b_i \in B$. Applying the averaging operators α we get $f^n + \alpha(b_1) f^{n-1} + \ldots + \alpha(b_0) = 0$ & use that $\alpha(b_i) \in B^G$ $\square$

Remark: There are other properties that A & A^G share. For example, we leave it as an exercise to show that for a finitely generated A equipped w. a rational representation of G by automorphisms, A is a domain (resp. reduced) iff so is A^G.

1.3) Application: normality of determinantal variety

Theorem from Sec 1.2 can be used to prove the normality of some varieties: a key point is that the algebra A^G is often easy to understand. As an example, take $m, n \in \mathbb{Z}_{\geq 0}$ & $r \leq \min(m,n)$. Consider the "determinantal variety" $X_r = \{A \in Mat_{m \times n} \mid rk\, A \leq r\}$ cf. Problem 2 in HW3.

Theorem: X_r is normal.

Proof:

The group $G = GL_m \times GL_n$ acts on X_r via $(g,h)A = gAh^{-1}$. Let

$$U = \left\{ \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \right\} \subset G, \text{ maximal unipotent.}$$

Let $F_i(A)$, $i = 1, \ldots r$, be the minor of size i in the top left corner of $A \in Mat_{m \times n}$, so $F_i \in \mathbb{C}[X_r]$.

Exercise: F_i is U-invariant.

We claim that $F_1, \ldots F_r$ are algebraically independent & generate $\mathbb{C}[X_r]^U$.

For this consider the subvariety $Y = \left\{ \begin{pmatrix} y_1 & & 0 & 0 \\ & \ddots & & \\ 0 & & y_r & \\ & 0 & & 0 \end{pmatrix} \middle| \, y_i \in \mathbb{C} \right\} \subset X_r$.

Using Gaussian elimination, one can show that the action map $U \times Y \xrightarrow{a} X_r$ is dominant, exercise. So

$$a^* : F \mapsto F|_Y : \mathbb{C}[X_r]^U \hookrightarrow \mathbb{C}[y_1, \ldots y_r].$$

In particular, $\mathbb{C}[X_r]^U$ is a domain, so $\mathbb{C}[X_r]$ is a domain, see Remark in Sec 1.2.

Note that $a^*(F_i) = G_i := y_1 \ldots y_i$, algebraically independent. It remains to show that $a^*(f) \in \mathbb{C}[G_1, \ldots G_r]$ $\forall f \in \mathbb{C}[X_r]^U$. Note that $\mathbb{C}[y_1, \ldots y_r] \subset \mathbb{C}[G_1^{\pm 1}, \ldots, G_r^{\pm 1}]$.

Recall that $\mathbb{C}[X_r]^U$ is graded by $\mathfrak{X}^+(T)$. We will only care about the grading by $\mathfrak{X}(T_m) = \mathbb{Z}^m$ for $T_m = \{ diag(*, \ldots *) \} \subset GL_m$.

Note that T acts on Y by $\mathrm{diag}(t_1, \ldots t_m).(y_1, \ldots y_r) = (t_1 y_1, \ldots t_r y_r)$ & a is equivariant. Let $\varepsilon_1, \ldots \varepsilon_m \in \mathfrak{X}(T_m)$ be the standard basis: ε_i sends $\mathrm{diag}(t_1, \ldots t_m)$ to t_i. Then y_i has degree $-\varepsilon_i$.

So, it's enough to consider a homogeneous element $f \Rightarrow a^*(f)$ is Laurent monomial in $G_1, \ldots G_r$ b/c different monomials have different degrees. But if a Laurent monomial $F_1^{d_1} \ldots F_r^{d_r} \in \mathbb{C}[X_r]$, then we must have $d_i \geq 0$: in order to see this we evaluate the monomial of $\begin{pmatrix} S_i & 0 \\ 0 & 0 \end{pmatrix}$, where S_i is a transposition matrix $\begin{pmatrix} 1 & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & 1 \end{pmatrix}$ $\leftarrow i$ where $F_j = \pm 1$ for $j \neq i$ & $F_i = 0$ & on $\begin{pmatrix} 1 & & 0 \\ & 1_0 & \\ 0 & & 0 \end{pmatrix}$ where $F_j = 1$ for $j < r$ & $F_r = 0$. We see that the monomials in the image of a^* are exactly $G_1^{d_1} \ldots G_r^{d_r}$ w. $d_i \geq 0$, which shows $a^*: \mathbb{C}[X_r]^U \xrightarrow{\sim} \mathbb{C}[G_1, \ldots G_r]$ finishing the proof. $\square$