Invariant theory 3, 1/22/25

1) Categorical quotients

References: [PV], Secs 4.3, 4.4, 1.2.

1.0) Recap & goals

In Sec 1.3 of Lec 2 we have introduced reductive algebraic groups (over $\mathbb{C}$); $GL_n(\mathbb{C}), SL_n(\mathbb{C}), SO_n(\mathbb{C}), Sp_{2n}(\mathbb{C})$ provide examples. We have seen that for the class of (finite dimensional) rational representations there is an averaging operator in the sense of Sec 1.1 of Lec 2, i.e. a collection (α_V), where V runs over the rational representations s.t.

(a1): $\operatorname{im} \alpha_V \subset V^G$,

(a2): $\alpha_V(v) = v \;\; \forall\, v \in V^G$

(a3): α is functorial: $\varphi \circ \alpha_V = \alpha_{V'} \circ \varphi \;\; \forall\, \varphi \in \operatorname{Hom}_G(V, V')$.

The existance of α implies:

Thm (Hilbert): Let G be reductive & V be a rational finite dimensional representation of G. Then $\mathbb{C}[V]^G$ is finitely generated.

In this lecture we will be concerned w. a more general situation. Let $\mathbb{F}$ be an algebraically closed (for simplicity) field & X be an affine variety over $\mathbb{F}$. Suppose that an algebraic group G acts on X in an algebraic way (i.e. the action map $a: G\times X \to X$ is a morphism). The action is by (auto)morphisms of X, hence it gives rise to an action of G on $\mathbb{F}[X]$ by algebra automorphisms.

Proposition 1: Suppose $\mathbb{F}=\mathbb{C}$ & G is reductive. Then $\mathbb{F}[X]^G$ is finitely generated.

As discussed in Sec 3 of Lec 1, we can form the variety, $X/\!\!/G$, the categorical quotient for $G \curvearrowright X$ - later in this lecture we will justify the name - together with a dominant morphism $\pi: X \to X/\!\!/G$. We have stated:

Proposition 2: π is surjective & every fiber contains a unique closed G-orbit.

The main goal of this lecture is to prove Propositions 1&2 & related properties.

1.1) General rational representations.

Let G be an algebraic group over $\mathbb{F}$. We can generalize the notion of rational to not necessarily finite dimensional representations as follows:

Definition: Let $\tilde{V}$ be a representation of G. We say that $\tilde{V}$ is *rational* if $\forall v \in \tilde{V}$ $\exists$ finite dimensional subrepresentation $V_0 \subset \tilde{V}$ w. $v \in V_0$, which is rational (as a G-representation).

Example: Let V be a finite dimensional rational representation. Step 3 of the proof of Thm in Sec 1.2 of Lec 2 shows that $\mathbb{F}[V]$ is rational.

The following generalization of this Example is a crucial ingredient in proving the propositions from Sec 1.0.

Proposition: Let X be an affine variety (or a finite type affine scheme over $\mathbb{F}$) w. a G-action. Then $\mathbb{F}[X]$ is a rational G-representation.

Proof: The action morphism $a: G \times X \to X$ gives the pullback

$a^*: \mathbb{F}[X] \longrightarrow \mathbb{F}[G\times X] \xrightarrow{\sim} \mathbb{F}[G]\otimes\mathbb{F}[X]$. Take $f\in\mathbb{F}[X]$. Then we can find linearly independent $f_1,\dots f_k\in\mathbb{F}[X]$, $h_1,\dots,h_k\in\mathbb{F}[G]$ w. $a^*f=\sum_{i=1}^k f_i\otimes h_i \Leftrightarrow f(gx)=\sum_{i=1}^k f_i(x)h_i(g)$ $\forall g\in G, x\in X$. In particular, for any g, we have $g^{-1}.f\in \operatorname{Span}_{\mathbb{F}}(f_i)$ (b/c $[g^{-1}.f](x)=f(gx)$). So f lies in a finite dimensional subrepresentation. We need to show it's rational. Set $V_0:=\operatorname{Span}_{\mathbb{F}}(f_i)$

First, we claim $\operatorname{Span}(g.f\mid g\in G)=V_0$. Indeed, since $h_1,\dots,h_k\in\mathbb{F}[G]$ are linearly independent $\exists\, g_1,\dots g_k\in G$ s.t. the vectors $(h_i(g_1),\dots,h_i(g_k))\in\mathbb{F}^k$, $i=1,\dots,k$, are linearly independent. From here we see that $V_0=\operatorname{Span}_{\mathbb{F}}(g_i^{-1}.f)$, yielding the claim.

Now we need to show V_0 is a rationel representation, equivalently, metrix coefficient $[g\mapsto\langle\beta,g.v\rangle]\in\mathbb{F}[G]$ $\forall \beta\in V_0^*, v\in V_0$. We can pick $v:=f$ b/c that function was chosen arbitrarily. Then

$$\langle\beta,g.v\rangle=\sum_{i=1}^k\langle\beta,f_i\rangle h_i(g^{-1})$$

giving a polynomial function on G & finishing the proof $\square$

We will need the following property. If $\varphi: U\to V$ is a homomorphism of rational representations, then φ restricts to a linear map $U^G\longrightarrow V^G$.

Lemma: Assume G is reductive (and $\mathbb{F}=\mathbb{C}$). If φ is surjective, then $\varphi(U^G) = V^G$.

Proof: If U, V are finite dimensional, then this follows from the existence of averaging operator – exercise (use (a2) & (a3)). In the general case, pick $v \in V^G$ & $u \in \varphi^{-1}(v)$. By definition, $\exists$ finite dimensional rational $U_0 \subset U$ w. $u \in U_0$. Now use the finite dimensional case for $\varphi|_{U_0}: U_0 \to \varphi(U_0)$ to see that $v \in \varphi(U_0^G)$. $\square$

Here's an actually stronger claim

Exercise: Suppose that G is an algebraic group over $\mathbb{F}$ s.t. the class of (finite dimensional) rational representations admits an averaging operator, α. Show that α uniquely extends to all rational representations so that (a1)-(a3) continue to hold.

1.2) Proof of Proposition 1

We will use Proposition from Sec 1.1 to realize $\mathbb{C}[X]$ as a G-equivariant algebra quotient of $\mathbb{C}[V]$ for suitable (finite dimensional) rational G-representation V and then use Lemma in Sec 1.2 & Hilbert's thm to finish the proof.

Step 1: Let $f_1, \ldots f_k \in \mathbb{C}[X]$ be algebra generators. By Prop'n in Sec 1.1 $\exists$ fin. dim. rational G-subrepresentation $V_i' \subset \mathbb{C}[X]$ w. $f_i \in V_i'$. Set $V' := \sum_{i=1}^{k} V'$. The inclusion $V' \hookrightarrow \mathbb{C}[X]$ gives rise to a G-equivariant algebra homomorphism

(1) $\quad \mathbb{C}[V'^*] = S(V') \longrightarrow \mathbb{C}[X]$.

Since $f_1, \ldots f_k \in V'$, (1) is surjective. And V' is rational (as a quotient of a rational representation $\bigoplus_{i=1}^{k} V_i'$). Now take $V := V'^*$.

Step 2: Apply Lemma from Sec 1.2 to $\varphi: \mathbb{C}[V] \twoheadrightarrow \mathbb{C}[X]$ from Step 1. The restriction $\varphi: \mathbb{C}[V]^G \twoheadrightarrow \mathbb{C}[X]^G$ is still an algebra homomorphism. By Hilbert's thm from Sec 1.2 in Lec 2, $\mathbb{C}[V]^G$ is finitely generated. Hence so is its quotient $\mathbb{C}[X]^G$ $\square$

Rem: Step 1 (that geometrically means: $X \hookrightarrow V$ G-equivariantly) doesn't need G to be reductive.

1.3) Universal property

Consider the morphism $\pi: X \to X/\!\!/G$. The following lemma justifies the name "categorical quotient."

Lemma: Let G be a reductive group acting on an affine variety X. Let Y be another affine variety & $\varphi: X \to Y$ be a G-invariant morphism. Then $\exists!\ \underline{\varphi}: X/\!\!/G \to Y$ w.

(2) $$\varphi = \underline{\varphi} \circ \pi.$$

Proof:

φ is G-invariant $\Leftrightarrow$ $h \circ \varphi \in \mathbb{C}[X]^G\ \forall\ h \in \mathbb{C}[Y]$ (to see $\Leftarrow$ let $(h_1, \dots h_n): Y \hookrightarrow \mathbb{A}^n$ and use $h := h_i$, $i = 1, \dots n$) $\Leftrightarrow$ $\operatorname{im} \varphi^* \subset \mathbb{C}[X]^G$. Set $\underline{\varphi}: X/\!\!/G \to Y$ to be the dual morphism to $\varphi^*: \mathbb{C}[Y] \to \mathbb{C}[X]^G$. Then $\varphi^* = \pi^* \circ \underline{\varphi}^* \Leftrightarrow (2)$. The uniqueness of $\underline{\varphi}$ satisfying (2) is an exercise. $\square$

Exercise 2: Let X', X be two affine varieties w. G-actions. Let $\varphi: X' \to X$ be a G-equivariant morphism. Then $\exists!$ $\underline{\varphi}: X'/\!\!/G \to X/\!\!/G$ making the following diagram commutative:

(*) $$\begin{tikzcd} X' \arrow[r, "\varphi"] \arrow[d, "\pi'"] & X \arrow[d, "\pi"] \\ X'/\!\!/G \arrow[r, "\underline{\varphi}"] & X/\!\!/G \end{tikzcd}$$

Moreover, $\underline{\varphi}^*: \mathbb{C}[X]^G \to \mathbb{C}[X']^G$ is the restriction of φ^*.

1.4) Further properties of π

As before, in this section G is a reductive group acting on an affine variety X. The following implies Prop 2 from Sec 1.2.

Proposition: i) π is surjective

ii) Let $X' \subset X$ be a closed G-stable subvariety & $\varphi: X' \hookrightarrow X$ denote the inclusion. Then $\underline{\varphi}$ from Exercise 2 in Sec 1.3 is a closed inclusion & it identifies $X'/\!/G$ w. $\pi(X')$.

iii) Let $X_1', X_2' \subset X$ be G-stable closed subvarieties w. $X_1' \cap X_2' = \emptyset$. Then $\pi(X_1') \cap \pi(X_2') = \emptyset$.

Proof: i) Let $x \in X/\!/G$ & $\mathfrak{m} \subset \mathbb{C}[X]^G$ be its maximal ideal. The claim that $\pi^{-1}(x) \neq \emptyset$ is equivalent to

(3) $\mathbb{C}[X]\mathfrak{m} (= \operatorname{Span}_{\mathbb{C}[X]}(\mathfrak{m})) \neq \mathbb{C}[X]$.

Suppose (3) fails: $\exists\, f_1, \dots f_k \in \mathfrak{m}$ & $h_1, \dots h_k \in \mathbb{C}[X]$ s.t.

(3') $\sum_{i=1}^{k} f_i h_i = 1$.

We want to apply the averaging operator to (3') - but first we need to construct it. Choose a rational representation V as in Sec 1.2 so that $\mathbb{C}[V] \twoheadrightarrow \mathbb{C}[X]$ G-equivariantly. Let $\mathbb{C}[V]_{\leq i}$ be the space of elements of $\deg \leq i$ & $\mathbb{C}[X]_{\leq i}$ be its image in $\mathbb{C}[X]$ so that $\mathbb{C}[X] = \bigcup_{i \geq 0} \mathbb{C}[X]_{\leq i}$. Now we can argue as in Step 3 of the proof of Hilbert's theorem (Sec 1.2 of Lec 2): $\mathbb{C}[X]_{\leq i}$ is a rational representation so comes with averaging operator, α_i. Then we define $\alpha: \mathbb{C}[X] \to \mathbb{C}[X]$ by: $\alpha(f) = \alpha_i(f)$ for $f \in \mathbb{C}[X]_{\leq i}$.

Apply α to (3'). Arguing as in Steps 4, 5 of the proof of Hilbert's thm, we get

$1 = \alpha(\sum f_i h_i) = \sum f_i \, \alpha(h_i)$. Since $\alpha(h_i) \in \mathbb{C}[X]^G$, this implies $1 \in \mathfrak{m}$ leading to contradiction.

ii) $\varphi^*: \mathbb{C}[X] \twoheadrightarrow \mathbb{C}[X']$ is G-equivariant. By Lemma in Sec 1.1, φ^* restricts to $\mathbb{C}[X]^G \twoheadrightarrow \mathbb{C}[X']^G$. So $\underline{\varphi}: X'/\!/G \to X/\!/G$ is a closed embedding. In diagram (*), π' is surjective by (i), so $\pi(X') = \operatorname{im} \underline{\varphi} \xleftarrow{\sim} X'/\!/G$ proving the claim.

iii) Set $X' = X_1' \sqcup X_2'$. Note that $\mathbb{C}[X'] = \mathbb{C}[X_1'] \oplus \mathbb{C}[X_2']$ w. diagonal G-action $\Rightarrow \mathbb{C}[X']^G = \mathbb{C}[X_1']^G \oplus \mathbb{C}[X_2']^G \Leftrightarrow X'/\!/G \xrightarrow{\sim} X_1'/\!/G \sqcup X_2'/\!/G$. Now iii) follows from ii) (applied to the inclusions $X_1', X_2', X' \hookrightarrow X$).

$\square$

Corollary: Every fiber of π contains a unique (Zariski) closed G-orbit.

Proof: $\forall\, x \in X/\!/G$, $\pi^{-1}(x)$ is a non-empty (by i) & G-stable closed subvariety. Any orbit of minimal dimension is closed, so there's at least one. If there are two distinct closed orbits X_1' & X_2', then $X_1' \cap X_2' = \emptyset$. Then $\pi(X_i') = x$ contradicting iii) $\square$

In particular, $X /\!/ G$ indeed parameterizes the closed G-orbits in X.

Exercise: If G is finite, then every fiber of π is a single G-orbit.