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1) Properties of quotients
2) More on reductive groups.

Refs: see below

1) Properties of quotients

Ref: [PV], Sec 3.9.

Our main question in this section is: suppose a variety X has some algebro-geometric property (normality/smoothness, etc.). Does the quotient $X/\!/G$ inherit this property?

The proof of the following lemma is an exercise.

Lemma 1: Let A be a commutative algebra & Γ be a group acting on A by automorphisms.

1) If A is reduced (= no nonzero nilpotents), then A^Γ is so.
2) If A is a domain, then A^Γ is so
3) If A is normal (= a domain integrally closed in its field of fractions) then A^Γ is so.

Before we proceed to further properties, we will discuss a (moral) application of 3) to computing the categorical quotients.

Suppose $\mathbb{F}=\mathbb{C}$ (as we'll see later, everything works if $\mathbb{F}$ is a general algebraically closed char 0 field). Let X be a normal affine variety acted on by a reductive group G. Let Y be another normal variety & $\varphi: X\to Y$ be a G-invariant morphism. By Lemma in Sec. 1.3 of Lec 3 (for Y affine) or Prob. 2 in HW1 φ factorizes as $\underline{\varphi}\circ\pi$, where $\pi: X\to X/\!/G$ is the natural morphism & $\underline{\varphi}: X/\!/G\to Y$.

Lemma 2: Suppose that every fiber of φ contains a unique closed G-orbit (in particular, nonempty $\Leftrightarrow$ φ is surjective) Then $\underline{\varphi}$ is an isomorphism.

Proof: is based on the following fact (that follows from the Zariski main theorem for quasi-finite morphisms (Lemma 37.43.3 in Stacks project):

Fact: Let $\mathbb{F}$ be algebraically closed & char 0. Let Y_1, Y_2 be varieties, Y_2 is normal & $\varphi: Y_1\to Y_2$ be a bijective morphism. Then φ is an isomorphism.

Exercise 1: Deduce the claim of the lemma from Fact (hint: use results from Lec 3 that say that $X/\!/G$ parameterizes the

Zariski closed orbits in X). $\square$

We will apply Lemma to compute $X/\!\!/G$ in a very basic case. Suppose that G is a Zariski closed subgroup in an algebraic group $\tilde{G}$. By §3.1.7 in [OV], $\tilde{G}/G$ admits a structure of a quasi-projective variety s.t. the natural action of $\tilde{G}$ on $\tilde{G}/G$ is algebraic. In fact, it's unique (w. conditions that $\tilde{G}$ acts transitively & the stabilizer of a point is G – this can also be deduced from Fact).

On the other hand, if G is reductive, we can form the categorical quotient $\tilde{G}/\!\!/G$ for the action of G on $\tilde{G}$ by right translations.

Corollary: We have an isomorphism of varieties $\tilde{G}/\!\!/G \xrightarrow{\sim} \tilde{G}/G$.

Proof:

In the setting of Lemma, take $X = \tilde{G}$ & $\varphi: \tilde{G} \longrightarrow \tilde{G}/G$, $\tilde{g} \mapsto \tilde{g}G$. Every fiber of φ is a single orbit, automatically closed. And, as any variety, $\tilde{G}/G$ has a smooth point. Thx to the transitive $\tilde{G}$-action, every point is smooth, hence $\tilde{G}/G$ is normal. Conditions of the lemma are satisfied implying the corollary. $\square$

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In particular, in the setting of Corollary, $\tilde{G}/G$ is affine. Later on we will see that if $\tilde{G}$ is reductive & $\tilde{G}/G$ is affine, then G is reductive. We will show this when $\mathbb{F}=\mathbb{C}$ using connections to Symplectic geometry.

We get back to our main topic.

Exercise 2: Suppose X is factorial (i.e $\mathbb{F}[X]$ is a UFD) & G is irreducible w/o nontrivial homomorphisms to the multiplicative group. Then $\mathbb{F}[X]^G$ is a UFD.

Example: The smoothness is generally not preserved. The simplest example is when $X=\mathbb{C}^2$ & $G=\{\pm 1\}$ acts by scaling. Then $\mathbb{C}[X]^G=\mathbb{C}[x^2, xy, y^2]\subset\mathbb{C}[x,y]$ is isomorphic to $\mathbb{C}[a,b,c]/(b^2-ac)$ the algebra of functions on a singular surface.

Bonus remark: Here's a nice property of singularities inherited by categorical quotients. Suppose $\mathbb{F}$ is an algebraically closed field of characteristic 0. A normal affine variety X is said to have rational singularities if $\exists$ ($\Leftrightarrow\forall$) resolution of singularities $\tilde{X}$ of X w. $H^i(\tilde{X}, \mathcal{O}_{\tilde{X}})=0$ $\forall$ $i>0$. By a theorem of

Boutot (1987), if G is a reductive group acting on a normal affine variety X w. rational singularities, then $X//G$ has rational singularities.

2) More on reductive groups.

We defined reductive groups over $\mathbb{C}$. The goal of this section is to do this over an arbitrary algebraically closed field $\mathbb{F}$. We will also clarify a connection with complete reducibility & averaging operators & discuss the behavior of categorical quotients.

2.1) Unipotent groups ([OV], §3.3.6; [Hu], Sec 17.5)

Let G denote an algebraic group.

Recall that, for a finite dimensional vector space V, a linear operator $A: V \to V$ is called unipotent if $A - id_V$ is nilpotent.

Proposition: TFAE

1) $\forall$ representation $\rho: G \to GL(V)$, $\rho(G)$ consists of unipotent elements.

2) $\exists$ faithful representation $\rho: G \to GL(V)$ s.t. $\rho(G)$ consists of unipotent elements

3) $\exists$ normal subgroups $G = G_0 \supset G_1 \supset G_2 \supset \ldots \supset G_k = \{1\}$ s.t. G_{i-1}/G_i

is isomorphic to the additive group $\mathbb{F}$ $\forall i=1,\dots k$.

Example: $G:=\left\{\begin{pmatrix} 1 & & * \\ & 1 & \\ 0 & & \ddots & 1\end{pmatrix}\right\}\subset GL_n(\mathbb{F})$ is unipotent: it manifestly satisfies 2) and is easily seen to satisfy 3).

Lemma/definition ([OV], §6.4; [Hu], Sec 6.4)
Let G be an algebraic group. $\exists!$ maximal (w.r.t. $\subseteq$) normal unipotent subgroup of G (called the unipotent radical of G & denoted by $R_u(G)$).

Example: Let G be the subgroup of block upper triangular matrices: $G=\left\{\begin{pmatrix} * & * & * \\ & * & * \\ 0 & & * \end{pmatrix}\right\}$. Then $R_u(G)=\left\{\begin{pmatrix} I & * & * \\ & I & * \\ 0 & & I\end{pmatrix}\right\}$.

2.2) Reductive groups ([Hu], Secs 8-10; [OV], Sec 4.2)

Here is the most common definition of a reductive group that works in any characteristic.

Def: We say an algebraic group G is reductive if $R_u(G)=\{1\}$.

One can show that G is reductive iff the connected component of 1 is reductive. We also have a characterization of connected reductive groups as follows.

Definition: • A connected reductive group is called simple if all of its normal subgroups are finite & it's nonabelian.

• A connected algebraic group is called semisimple if it's a (not necessarily direct) product of simple normal subgroups.

• By a torus we mean the direct product of several copies of the multiplicative group $\mathbb{F}^\times$.

Proposition: Let G be a connected algebraic group. TFAE:

(1) G is reductive

(2) G is isomorphic to a (not necessarily direct) product of a semisimple group and a torus.

Example: the groups $SL_n(\mathbb{F})$ & $Sp_{2n}(\mathbb{F})$ are simple for all n. The groups $SO_n(\mathbb{F})$ are simple for $n=3$ or $n \geq 5$. For $n=2$, $SO_2(\mathbb{F}) \simeq \mathbb{F}^\times$, while $SO_4(\mathbb{F})$ is the product of two copies of $SL_2(\mathbb{F})$ (intersecting at their centers $\{\pm 1\}$). The group $GL_n(\mathbb{F})$ is the product of $SL_n(\mathbb{F})$ and the subgroup of scalar matrices, $\{\mathrm{diag}(z, z, \dots, z) \mid z \in \mathbb{F}^\times\}$

Using the proposition and other classification results one shows that over $\mathbb{C}$ the definition of a reductive group in this section is equivalent to one given in Sec 1.3 of Lec 2.

2.3) Complete reducibility ([N])

Definition: Let $\mathbb{F}$ be an algebraically closed field and G be an algebraic group over $\mathbb{F}$. We say that G is *linearly reductive* if any (finite dimensional, equivalently, arbitrary) rational representation of G is completely reducible.

Exercise 1: Show that the following are equivalent:

(a) Any finite dimensional rational representation is completely reducible

(b) The class of finite dimensional rational representations admits an averaging operator.

Further, show that if (a) holds, then $\exists!$ G-equivariant averaging operator.

Hints: (b) $\Rightarrow$ (a): for a representation V & a subrepresentation $U \subset V$ look at $\operatorname{Hom}_{\mathbb{F}}(V,U) \twoheadrightarrow \operatorname{Hom}_{\mathbb{F}}(U,U)$

A connection between the two kinds of reductivity is as follows.

Theorem: Assume $\mathbb{F}$ is of characteristic 0. TFAE:

(i) G is reductive

(ii) G is linearly reductive.

For $\mathbb{F}=\mathbb{C}$, we have briefly discussed that (i) implies our initial definition of reductive which implies linearly reductive thx. to Exercise 1. In the general case an argument is trickier: the most essential ingredient is the complete reducibility of finite dimensional representations of semisimple Lie algebras.

The implication ii) $\Rightarrow$ i) works w/o restrictions on char $\mathbb{F}$ & follows from the next exercise.

Exercise 2: 1) Show that any algebraic group G admits a faithful finite dimensional rational representation (if we remove "finite dimensional", then the claim is easier: look at the regular representation $\mathbb{F}[G]$).

2) $R_u(G)$ acts by 1 on any completely reducible rational representation (hint: you need an algebraic analog of Engel's thm).

Notice that Theorem & Exercise 1 imply that if char $\mathbb{F}=0$, then Propositions 1&2 from Sec 1.0 in Lec 3 (as well as Proposition in Sec 1.4 of Lec 3) still hold. An interesting fact is that we can remove the condition of char $\mathbb{F}=0$, more on this in the next section.

Bonus remark: Here is another characterization of reductive groups due to V. Popov (1979): TFAE
- G is reductive
- R^G is finitely generated $\forall$ finitely generated commutative algebra R equipped w. rational G-representation by automorphisms.

2.4) Bonus: geometrically reductive groups.

Here we explain what happens in characteristic p. A reference for this section is [MF], Appendix to Chapter 1, A&C. Below $\mathbb{F}$ is an algebraically closed field & G is an algebraic group over $\mathbb{F}$.

A problem with char $\mathbb{F}=p$ is that there are too few linearly reductive groups: according to Nagata (1961) those are exactly G s.t. the connected component G° is a torus, while G/G° (a finite group) has order coprime to p.

Here's a condition weaker than the linear reductivity.

Definition: G is ***geometrically reductive*** if for any finite dimensional rational representation V and any $v \in V^G$ $\exists\, f \in \mathbb{F}[V]_i^G$ for $i > 0$ s.t. $f(v) \neq 0$.

Note that the condition of being linearly reductive is equivalent to the existence of f in $(V^*)^G$, i.e. for $i = 1$.

The following is a result of Haboush from 1975.

Theorem: G is reductive $\Leftrightarrow$ G is geometrically reductive.

It turns out (see [MF], C in Appendix to Chapter 1) that results of Lec 3 regarding the finite generation of $\mathbb{F}[X]^G$ & properties of $\pi : X \to X /\!/ G$ still hold for geometrically reductive groups.