Invariant theory 5, 1/27/25

1) θ-groups: motivation & basic definitions.
2) Some structural results.

Ref: [OV], Secs 3.3 & 4.4; [V]

1) θ-groups: motivation & basic definitions.

Recall that in Sec 2 of Lec 1 we have considered the following nice example of a rational representation: the group $G=GL_n(\mathbb{C})$ acts on $G=\mathfrak{gl}_n(\mathbb{C})$ by conjugations: $g.A=gAg^{-1}$. We have mentioned that

(a) The algebra $\mathbb{C}[V]^G$ is free $\Leftrightarrow$ $V/\!\!/G$ is an affine space.

(b) $\forall$ fiber of $\pi: V \twoheadrightarrow V/\!\!/G$ contains finitely many G-orbits.

Informally, the categorical quotient is easy – (a) – & does a good job parameterizing orbits – (b). One can ask about more general rational representations satisfying (a) & (b). The goal of this part of the course is to present a generalization of this example satisfying (a) & (b) – Vinberg's θ-groups.

Rem: Here are other favorable properties. The morphism π is flat. Moreover it has a section $\iota: V/\!/G \hookrightarrow V$: the matrix $\begin{pmatrix} 1 & & 0 & -a_0 \\ & \ddots & & -a_1 \\ & & & \vdots \\ 0 & & 1 & -a_{n-1} \end{pmatrix}$ has characteristic polynomial $\lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_0$.

1.1) Adjoint representation.

It turns out that for an arbitrary connected reductive (in particular, s/simple) algebraic group G the adjoint representation of G in $\mathfrak{g}$ has properties (a) & (b), moreover $\dim \mathfrak{g}/\!/G = \operatorname{rk} \mathfrak{g}$. It also has properties in the remark.

There are several reasons to care about the adjoint representations coming from Representation theory & other subjects (such as the study of integrable systems or Algebraic geometry). The role in Representation theory is as follows:

- The study of various aspects of this action is important for the geometric construction of representations of Weyl groups & affine Hecke algebras. [CG] is about this.
- The closely related adjoint action of G on itself plays an important role in understanding of representations of finite groups of Lie type.
- The adjoint representation plays a crucial role in the study of certain infinite dimensional representations of $\mathfrak{g}$

(such as category $\mathcal{O}$) & G.

1.2) Automorphisms/gradings.

Let G be a connected reductive algebraic group over $\mathbb{C}$, and $\mathfrak{g}$ be its Lie algebra. Fix $d>1$ & consider an order d automorphism θ of G. It gives rise to an order d automorphism of $\mathfrak{g}$. Once we fix a primitive root of 1, ε (e.g. $\varepsilon=\exp(2\pi\sqrt{-1}/d)$) the data of θ can be interpreted as a Lie algebra grading $\mathfrak{g}=\bigoplus_{i\in\mathbb{Z}/d\mathbb{Z}}\mathfrak{g}_i$ (meaning $[\mathfrak{g}_i,\mathfrak{g}_j]\subset\mathfrak{g}_{i+j}$): $\mathfrak{g}_i=\ker(\theta-\varepsilon^i)$. Conversely, given a grading $\mathfrak{g}=\bigoplus_i\mathfrak{g}_i$, we can produce an automorphism of $\mathfrak{g}$. We can uniquely lift θ to an order d automorphism of the simply connected (automatically algebraic) semisimple group G w. Lie algebra $\mathfrak{g}$. Note that the restriction of $Ad: G\to GL(\mathfrak{g})$ to G^θ fixes all $\mathfrak{g}_i$ (exercise).

Definition: By a θ-group we mean G^θ with its linear action of $\mathfrak{g}_1$.

Examples: 1) Let $\underline{G}$ be a connected reductive algebraic group & set $G=\underline{G}\times\underline{G}$. Consider $\theta: G\to G$ given by $(g_1,g_2)\mapsto(g_2,g_1)$. Then $G^\theta=\underline{G}$ embedded diagonally & $\mathfrak{g}_1=\{(x,-x)\,|\,x\in\underline{\mathfrak{g}}\}$

viewed as a representation of $\underline{G}$. So the adjoint representation is a special case.

2) Let $G=GL_n$, $d=2$ & $\theta(g)=(g^T)^{-1}$. Then $G=O_n$ & $\mathfrak{g}_1=\{\text{symmetric matrices}\}\underset{G}{\simeq} S^2(\mathbb{C}^n)$.

Similarly, we can get $G^\theta=Sp_n$ (for even n) & $\mathfrak{g}_1\simeq\Lambda^2(\mathbb{C}^n)$.

3) Let $G=GL_n$, d be arbitrary & θ is given by conjugation with an order d element in GL_n. In a suitable basis such an element g takes the form $\mathrm{diag}(\overbrace{1,\dots,1}^{n_0},\overbrace{\varepsilon,\dots,\varepsilon}^{n_1},\dots,\varepsilon^{d-1},\dots,\varepsilon^{d-1})$. Let V_i denote the eigenspace for g with eigenvalue ε^i. Then

$$G^\theta=\prod_{i=0}^{d-1}GL(V_i)$$

& $\mathfrak{g}_1=\bigoplus_{i=0}^{d-1}\mathrm{Hom}(V_i,V_{i+1})$, where $V_d:=V_0$. In other words, we get the representation space for affine Dynkin quiver $\tilde{A}_n$ with cyclic quiver.

Note $G^\theta\subset G$ is an algebraic subgroup with Lie algebra $\mathfrak{g}^\theta$. Here are further properties.

Proposition: 1) G^θ is reductive

2) If G is not a torus, then $\dim G^\theta>1$

3) If G is semisimple & simply connected, G^θ is connected.

Sketch of proof: 2) for semisimple G is Thm 2 in §4.4.2, [OV]. In general, (G,G) is an algebraic subgroup of G & it's semisimple & connected. It's preserved by θ. If $(G,G)=\{1\}$, then G is a torus.

3) is Thm 9 in §4.4.8, [OV].

We prove 1). Assume the contrary: $R_u(G^\theta)\neq\{1\}$. Note that $G/(G,G)$ is a commutative connected reductive group hence a torus. It admits no nontrivial homomorphisms from a unipotent group. So $R_u(G^\theta)\subset(G,G)$. We replace G w. (G,G) & assume G is semisimple.

Let $\mathfrak{n}$ be the Lie algebra of $R_u(G^\theta)$. Since $R_u(G^\theta)\subset G^\theta$ is normal, $\mathfrak{n}$ is an ideal in $\mathfrak{g}^\theta=\mathfrak{g}_0$. Since $R_u(G^\theta)$ is unipotent, it acts by unipotent operators on $\mathfrak{g}$. So, $ad(x):\mathfrak{g}\to\mathfrak{g}$ is nilpotent $\forall x\in\mathfrak{n}$. Now we are done by the following two claims that are left as exercises.

1) For the Killing form $(\cdot,\cdot)$ on $\mathfrak{g}$, we have $(\mathfrak{g}_i,\mathfrak{g}_j)=0$ for $i+j\neq 0$. In particular, $(\cdot,\cdot)|_{\mathfrak{g}_0}$ is nondegenerate.

2) Let $\mathfrak{g}'\subset\mathfrak{g}$ be a subalgebra and $\mathfrak{n}\subset\mathfrak{g}'$ be an ideal s.t. $ad(x):\mathfrak{g}\to\mathfrak{g}$ is nilpotent $\forall\, x\in\mathfrak{g}'$. Then $(\mathfrak{n},\mathfrak{g}')=0$ (hint: Engel's thm).

So in our situation $\mathfrak{n}=\{0\}\Rightarrow R_u(G)=\{1\}$. $\square$

Let's explain some motivations to consider θ-groups. The case $d=2$ is very classical and goes back to the work of E. Cartan in the 1920's. Various features of this case are closely related to the study of several aspects of real semisimple Lie groups (cf. David Nadler's talk at the U. Minnesota relative Langlands workshop). Another (related) motivation is the study of symmetric spaces (whose algebraic incarnations are the homogeneous spaces G/G^θ). And the case of $d>2$ allows to understand some very classical examples of linear actions. For example, SL_3 acting on $S^3(\mathbb{C}^3)$ (studied first by Poincare in 1880's) arises from an order 3 automorphism.

As a spoiler, we will see that many invariant theoretic features of θ-groups mirror the more familiar story of adjoint representations (Cartan spaces, Weyl groups, etc.). Some new features appear for $d>2$, most notably, the Weyl group is sometimes a <u>complex</u> reflection group (not a crystallographic a.k.a. rational one as in the usual one).

Remark: The case of action of G^θ on $\mathfrak{g}_i$ for $i\neq 0$ reduces to $i=1$: for $e=GCD(i,d)$. We replace G w. $G^{(\theta^e)}$ & θ w. its suitable power coprime to d.

2) Some structural results.

2.1) Semisimple & nilpotent elements.

A reference: [OV], §3.3.7. Let G be a connected algebraic group over $\mathbb{C}$ & $\mathfrak{g}$ be its Lie algebra. Abusing the terminology, by a rational representation of $\mathfrak{g}$ we mean the differential of a finite dimensional rational representation of G.

Definition/Fact: Let $x \in \mathfrak{g}$. TFAE:

(1) $\forall$ rational representation $\rho: \mathfrak{g} \to \mathfrak{gl}(V)$ the element $\rho(x)$ is semisimple $\overset{\text{def}}{\Leftrightarrow}$ diagonalizable (resp. nilpotent)

(2) the same for some faithful representation.

In this case we call x semisimple (resp. nilpotent).

Thm (Jordan decomposition)

$\forall x \in \mathfrak{g}$ $\exists!$ x_s, x_n (the semisimple & nilpotent parts of x) s.t.

1) x_s is semisimple & x_n is nilpotent.

2) $x = x_s + x_n$.

3) $[x_s, x_n] = 0$.

Example: Let $\mathfrak{g} = \mathfrak{sl}_n$; $\forall x \in \mathfrak{g}$, $\exists$ basis s.t. x is given by a Jordan matrix. Then x_s is the diagonal part & x_n is the part above the diagonal.

Exercise: Let $\Phi: G \to H$ be an algebraic group homomorphism, and $\varphi = d_1\Phi$. Then $\varphi(x_s) = \varphi(x)_s$, $\varphi(x)_n = \varphi(x_n)$.

Corollary: Let $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}/d\mathbb{Z}} \mathfrak{g}_i$ be a grading & $x \in \mathfrak{g}_i$ for some i that comes from an order d automorphism of G (this is the case when $\mathfrak{g}$ is semisimple for example). Then $x_s, x_n \in \mathfrak{g}_i$.

Proof: Applying exercise to θ we get $\theta(x_s) = \theta(x)_s$. But $\theta(x) = \varepsilon^i x \Rightarrow \theta(x_s) = \varepsilon^i x_s \Rightarrow x_s \in \mathfrak{g}_i \Rightarrow x_n \in \mathfrak{g}_i$ $\square$

2.2) Cartan subspace

Definition (Cartan/Vinberg) By a Cartan subspace in $\mathfrak{g}$, we mean a maximal (w.r.t. $\subseteq$) subspace consisting of pairwise commuting semisimple elements.

Example: 1) For $G \curvearrowright \mathfrak{g}$ ($\xrightarrow{\sim} \{(x,-x) \mid x \in \mathfrak{g}\}$) we recover a definition of a Cartan subalgebra.

2) Let $\mathfrak{g} = \mathfrak{sl}_2$ & $\theta = Ad(diag(z, z^{-1}))$, where z is a primitive 3rd root of 1. Then $d = 3$, G^θ is the diagonal matrices & $\mathfrak{g}_1 = \{\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}\}$. In particular, the Cartan space here is zero.