Invariant theory 6

1) Cartan spaces for θ-groups
2) Weyl groups for θ-groups
3) Vinberg's lemma & applications.

Ref: [V].

1) Cartan spaces for θ-groups

1.0) Reminder (from Lec 5)

The base field is $\mathbb{C}$. Let G be a connected reductive group, θ an order d automorphism of G, $\mathfrak{g} = Lie(G)$. If we fix a primitive $\varepsilon = \sqrt[d]{1}$, then θ gives rise to a $\mathbb{Z}/d\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus_{i\in \mathbb{Z}/d\mathbb{Z}} \mathfrak{g}_i$.

As we have seen in Sec 1.2 of Lec 5, G^θ is a reductive group, of positive dimension if $(G,G) \neq \{1\}$. Let G_0 be the connected component of 1 in G^θ (we have $G_0 = G^\theta$ if G is simply connected). We are interested in the representation of G_0 in $\mathfrak{g}_1$. It turns out that this family of representations has many features of their special case: adjoint representations of semisimple groups.

For example, in Sec 2.2 we have introduced Cartan subspaces in $\mathfrak{g}_1$: maximal (w.r.t. $\subseteq$) subspaces consisting of pairwise commuting semisimple elements.

1.1) Conjugacy of Cartan subspaces

Any two Cartan subalgebras in $\mathfrak{g}$ are conjugate by an element of G. This generalizes to the graded setup.

Thm (Vinberg) Any two Cartan subspaces in $\mathfrak{g}_1$ are conjugate by an element of G_0.

Proof: Let $\mathfrak{a}^1, \mathfrak{a}^2$ be Cartan subspaces in $\mathfrak{g}_1$ & set $\mathfrak{z}^i := \{x \in \mathfrak{g}_1 \mid [x, \mathfrak{a}^i] = 0\}$, $i = 1,2$.

We will generalize the usual argument for the adjoint representations, in particular, show that $G_0 \mathfrak{z}^i$ is Zariski dense in $\mathfrak{g}_1$.

Step 1: We omit the superscript in this step: set $\mathfrak{a} := \mathfrak{a}^i$, $\mathfrak{z} := \mathfrak{z}^i$.

Take a Zariski generic $x \in \mathfrak{a}$, in particular,

(*) $\qquad \ker(\operatorname{ad} x) \cap \mathfrak{g}_1 = \mathfrak{z}$.

We claim that

(1) $\qquad \mathfrak{z} \oplus [\mathfrak{g}_0, x] = \mathfrak{g}_1$

Note that since x is semisimple, $[\mathfrak{g}, x]$ is the sum of eigenspaces for x with nonzero eigenvalues. So

(2) $\qquad \ker \operatorname{ad} x \oplus [\mathfrak{g}, x] = \mathfrak{g}$

We note that $[\mathfrak{g}_i, x] \subset \mathfrak{g}_{i+1}$, hence

(3) $\ker(\operatorname{ad} x) = \bigoplus_i [\ker(\operatorname{ad} x) \cap \mathfrak{g}_i]$, $[\mathfrak{g}, x] = \bigoplus_i [\mathfrak{g}_i, x]$.

Combining (2) and (3) yields (1).

Note that (1) implies that $G_0\mathfrak{z}^i$ contains a Zariski open subset in $\mathfrak{g}_1$.

Step 2: Our goal in this and further steps is to recover $\mathfrak{a}^i$ from a Zariski generic element of $\mathfrak{z}^i$. We again omit the superscript.

Note that all semisimple elements in $\mathfrak{z}$ are in $\mathfrak{a}$ (by the maximality condition). So $\mathfrak{a}$ can be recovered as the set of semisimple elements in $\ker(\mathrm{ad}\, x)\cap\mathfrak{g}_1$ for a Zariski generic element $x\in\mathfrak{a}$. Our question is therefore how to recover a Zariski generic element of $\mathfrak{a}$ from that of $\mathfrak{z}$. We'll see that the answer is: as the semisimple part.

Step 3: We claim that $z_s\in\mathfrak{a}$ $\forall z\in\mathfrak{z}$. Consider the subgroup $Z_G(\mathfrak{a})=\{g\in G \mid g.x=x\ \forall x\in\mathfrak{a}\}$. It's an algebraic subgroup w. Lie algebra $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a})=\{y\in\mathfrak{g} \mid [y,\mathfrak{a}]=0\}$. Note that $\mathfrak{z}=\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a})\cap\mathfrak{g}_1$. Then $z_s\in\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a})$ (exercise) & $z_s\in\mathfrak{g}_1$ by Corollary in Sec 2.1 of Lec 5. Hence $z_s\in\mathfrak{z}\Rightarrow z_s\in\mathfrak{a}$.

Step 4: Let $\mathcal{N}$ denote the subset of all nilpotent elements in $\mathfrak{z}$, a closed subvariety. Consider the morphism:

$$\mathfrak{a}\times\mathcal{N}\longrightarrow\mathfrak{z},\ (x,y)\mapsto x+y. \tag{4}$$

We claim that (4) is iso. Since $\mathfrak{z}$ is normal, it's sufficient to prove (4) is bijective (see Fact in Sec 1 of Lec 4). By Step 3, $(z_s, z_n) \in \mathfrak{a} \times \mathcal{N}$ $\forall$ $z \in \mathfrak{z}$. Then $z \mapsto (z_s, z_n)$ is an inverse of (4) (exercise).

Step 5: Since (4) is an isomorphism, $z \mapsto z_s : \mathfrak{z} \twoheadrightarrow \mathfrak{a}$ is a morphism. It follows from here and (*) that

$$\mathring{\mathfrak{z}} = \{ z \in \mathfrak{z} \mid \ker(\operatorname{ad} z_s) \cap \mathfrak{g}_1 = \mathfrak{z} \}$$

is Zariski dense (and open) in $\mathfrak{z}$

By Step 1, $G_0 \mathring{\mathfrak{z}}^i$, $i = 1, 2$, contain dense open subsets of $\mathfrak{g}_1$ $\Rightarrow G_0 \mathring{\mathfrak{z}}^1 \cap G_0 \mathring{\mathfrak{z}}^2 \neq \emptyset$. Replacing $\mathfrak{a}^2$ with $g.\mathfrak{a}^2$ for suitable $g \in G_0$ we can assume $\exists\, z \in \mathring{\mathfrak{z}}^1 \cap \mathring{\mathfrak{z}}^2$. We then recover $\mathfrak{a}^i$ from z as explained in Step 2 & get $\mathfrak{a}^1 = \mathfrak{a}^2$. $\square$

2) Weyl groups for θ-groups

Consider the subgroups

$$N_{G_0}(\mathfrak{a}) = \{ g \in G_0 \mid g\mathfrak{a} = \mathfrak{a} \} \supset Z_{G_0}(\mathfrak{a}) = \{ g \in N_{G_0}(\mathfrak{a}) \mid gx = x \ \forall x \in \mathfrak{a} \}$$

Note that $Z_{G_0}(\mathfrak{a}) \subset N_{G_0}(\mathfrak{a})$ is normal. Similarly, we get subgroups $Z_G(\mathfrak{a}) \triangleleft N_G(\mathfrak{a})$.

The quotient $N_{G_0}(\mathfrak{a}) / Z_{G_0}(\mathfrak{a})$ to be denoted by W_θ acts $\mathfrak{a}$ by linear transformations. It's called the Weyl group (of G_0 acting on $\mathfrak{g}_1$).

Lemma: W_θ is finite.

Proof: Note that $W_\theta = N_{G_0}(\mathfrak{a})/Z_{G_0}(\mathfrak{a}) \hookrightarrow N_G(\mathfrak{a})/Z_G(\mathfrak{a})$
So it's enough to show $N_G(\mathfrak{a})/Z_G(\mathfrak{a})$ is finite. This claim reduces to semisimple G (details are an exercise). In this case note that the weights of the representation of $\mathfrak{a}$ in $\mathfrak{g}$ span $\mathfrak{a}^*$. And $N_G(\mathfrak{a})$ preserves this finite set. An element fixing each individual weight must act trivially on $\mathfrak{a}^*$, hence on $\mathfrak{a}$, hence is in $Z_G(\mathfrak{a})$. $N_G(\mathfrak{a})/Z_G(\mathfrak{a})$ embeds into the permutations of a finite set, so is finite. $\square$

Our first important result re W_θ is a generalization of the Chevalley restriction theorem.

Thm (Chevalley/Vinberg): Let $\mathfrak{a} \subset \mathfrak{g}_1$ denote a Cartan subspace and $\iota: \mathfrak{a} \hookrightarrow \mathfrak{g}_1$ be the inclusion. The restriction of $\iota^*: \mathbb{C}[\mathfrak{g}_1] \twoheadrightarrow \mathbb{C}[\mathfrak{a}]$ to $\mathbb{C}[\mathfrak{g}_1]^{G_0}$ gives $\mathbb{C}[\mathfrak{g}_1]^{G_0} \xrightarrow{\sim} \mathbb{C}[\mathfrak{a}]^{W_\theta}$.

Exercise: $\iota^*(\mathbb{C}[\mathfrak{g}_1]^{G_0}) \subset \mathbb{C}[\mathfrak{a}]^{W_\theta}$

The proof (to be given in Lec 7) follows that of the usual Chevalley restriction theorem for $G \curvearrowright \mathfrak{g}$ but is more involved.
In Sec. 3 we will start developing tools for the proof.

3) Vinberg's lemma & applications.

3.1) Statement & proof.

Let G be a connected algebraic group, $H\subset G$ a connected algebraic subgroup, V a finite dimensional rational G-representation, $U\subset V$ an H-subrepresentation. Suppose we can find H-subrepresentations $\mathfrak{h}'\subset\mathfrak{g}$ & $U'\subset V$ s.t.

(i) $\mathfrak{h}\oplus\mathfrak{h}'=\mathfrak{g}$, $U\oplus U'=V$

(ii) $xu\in U'$ $\forall$ $x\in\mathfrak{h}'$, $u\in U$.

Examples: (i) is always achievable when H is reductive. In addition, we can achieve (ii) in the following two situations:

(a) $V=\mathfrak{g}$, $\mathfrak{h}'=U'$ is some H-stable complement of $\mathfrak{h}$

(b) $\mathfrak{g}=\bigoplus_{i\in\mathbb{Z}/d\mathbb{Z}}$ as before, $H=G_0$, $U=\mathfrak{g}_1$. Here we can take $\mathfrak{h}'=\bigoplus_{i\neq 0}\mathfrak{g}_i$ & $U'=\bigoplus_{i\neq 1}\mathfrak{g}_i$. To check (ii) is an exercise.

Theorem: Under assumptions (i), (ii) above the following holds: $\forall$ $u\in U$, the intersection $Gu\cap U$ is reduced & every connected (in Zariski topology) component of $Gu\cap U$ is a single H-orbit.

Proof:

$\forall v\in G.u\cap U\Rightarrow T_v(Gu\cap U)=[Gv=Gu]=\mathfrak{g}.v\cap U=[\mathfrak{h}.v\subset U,\ \mathfrak{h}'v\subset U']=\mathfrak{h}.v=T_vHv\Rightarrow$[$H$-equivariance] $T_{v'}(Gu\cap U)=T_{v'}Hv$ $\forall v'\in Hv$.

Since Hv is a smooth irreducible scheme, this shows Hv is open in $G \cap U$ & v is a reduced point. This implies our claim. $\square$

3.2) Applications

We will apply Theorem to study G_0-orbits of semisimple & nilpotent elements in $\mathfrak{g}_1$ (to be called semisimple & nilpotent orbits).

Proposition 1: Let $x \in \mathfrak{g}_1$ be semisimple. Then $G_0 x$ is closed.

Proposition 2: The number of nilpotent G_0-orbits in $\mathfrak{g}_1$ is finite.

The ideas of proofs for both propositions are similar: for $G = GL_n$ acting on $\mathfrak{gl}_n$ both claims are known or easy. Then we apply Theorem in Example a (with G becoming H there & GL_n becoming G) to deduce the claims for the adjoint representation of an arbitrary (connected reductive) G. Then we deduce the general case by applying Theorem in Example (6).

Remarks: 1) Prop 2 for the adjoint representation is usually proved using the Jacobson-Morozov theory

2) The converse of Proposition 1 is also true: if $G_0 x$ is closed, then x is semisimple, this will be deduced from Prop 2.

Proof of Prop. 1:

Step 1 ($GL_n \curvearrowright \mathfrak{gl}_n$): It's easy to deduce from the JNF theorem that $\overline{GL_n . x} \supset GL_n . x_s$ $\forall x \in \mathfrak{gl}_n$ (to do this is an exercise). To deduce the claim from this fact is also an exercise. So, $GL_n . x_s$ has minimal dimension in $\pi^{-1}(\pi(x_s))$ for $\pi: \mathfrak{gl}_n \twoheadrightarrow \mathfrak{gl}_n /\!/ GL_n$, the quotient morphism, hence $GL_n . x_s$ is closed.

Step 2 ($G \curvearrowright \mathfrak{g}$): By Exercise 2 in Sec 2.3 in Lec 4, G admits a faithful finite dimensional rational representation, V. By §3.1.4 in [OV], the image of an algebraic group homomorphism is closed. So we can view G as an algebraic subgroup of $GL(V)$.

Apply theorem in Example a (to $G \subset GL(V)$). We see that for semisimple $x \in \mathfrak{g}$, $GL(V).x$ is closed $\Rightarrow$ $GL(V).x \cap \mathfrak{g}$ is closed. But $G.x$ is a connected component in the closed subvariety $GL(V).x \cap \mathfrak{g}$, hence is closed.

Step 3: We argue as in the previous paragraph but applying Thm in Example 6). This finishes the proof, details are left as an exercise. $\square$

Proof of Prop 2: By the JNF theorem, there are finitely many nilpotent $GL(V)$-orbits in $\mathfrak{gl}(V)$. We argue as in the proof of

Prop 1 but use that a variety has only finitely many connected components. Details are left as an exercise. $\square$