Invariant theory 7, 02/03/05.

1) Proof of Chevalley restriction theorem
2) Weyl group is a complex reflection group.

Refs: [V]; [PV], Sec 8.3.

1.0) Reminder

We are in the setting of Sec 1.0 of Lec 6: G is a connected reductive group$/\mathbb{C}$ w. an order d automorphism θ giving rise to the grading $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}/d\mathbb{Z}} \mathfrak{g}_i$. We care about the action of the connected reductive group $G_0 = (G^\theta)^\circ$ on $\mathfrak{g}_1$.

In Lec 5 we have introduced Cartan subspaces $\mathfrak{a} \subset \mathfrak{g}_1$: maximal subspaces of pairwise commuting semisimple elements. Such a subspace is acted on by the finite group $W_\theta = N_{G_0}(\mathfrak{a})/Z_{G_0}(\mathfrak{a})$, the Weyl group. We have stated the following general version of the Chevalley restriction theorem due to Vinberg:

Thm: Let $\iota: \mathfrak{a} \hookrightarrow \mathfrak{g}_1$ denote the inclusion map. Then

$$\iota^*: \mathbb{C}[\mathfrak{g}_1]^{G_0} \xrightarrow{\sim} \mathbb{C}[\mathfrak{a}]^{W_\theta}$$

In Sec 3.2 we have shown two results useful to prove the theorem:

Proposition 1: If $x \in \mathfrak{g}_1$ is semisimple, then $G_0 x$ is closed.

Proposition 2: The number of nilpotent G_0-orbits in $\mathfrak{g}_1$ is finite.

1.1) Closed G_0-orbits in $\mathfrak{g}_1$

Proposition 3: We have $G_0 x_s \subset \overline{G_0 x}$ $\forall x \in \mathfrak{g}_1$. In particular, if $G_0 x$ is closed, then x is semisimple.

Proof: We first prove that $\overline{G_0 x} \ni 0$ for nilpotent x and then reduce to this case by passing to a suitable θ-stable subgroup of G.

Case 1: x is nilpotent. Let $\mathcal{N}$ denote the locus of nilpotent elements in $\mathfrak{g}_1$, a closed subvariety stable under the action of $G_0 \times \mathbb{C}^\times$, where $\mathbb{C}^\times$ acts by scaling. Since $\mathcal{N}/\!/G_0$ parameterizes the closed G_0-orbits in $\mathcal{N}$, our task is to show $\mathcal{N}/\!/G_0 = pt$. First, observe that $\mathbb{C}[\mathcal{N}]^{G_0} \subset \mathbb{C}[\mathcal{N}]$ is $\mathbb{C}^\times$-stable b/c the actions of G_0 and $\mathbb{C}^\times$ commute. We have the following diagram of algebra homomorphisms

$$\begin{array}{ccc} & \mathbb{C}[\mathfrak{g}_1]^{\mathbb{C}^\times} \xrightarrow{(1)} \mathbb{C}[\mathcal{N}]^{\mathbb{C}^\times} \\ & \qquad\qquad \cup \\ \mathbb{C}[\mathcal{N}/\!/G_0] \xleftarrow{(2)} & \mathbb{C}[\mathcal{N}/\!/G_0]^{\mathbb{C}^\times} = \mathbb{C}[\mathcal{N}]^{G_0 \times \mathbb{C}^\times} \end{array}$$

(1) is surjective by Proposition in Sec 1.4 in Lec 3. Note

that $\mathbb{C}[\mathfrak{g}_1]^{\mathbb{C}^\times}=\mathbb{C}$ b/c the $\mathbb{C}^\times$-action is by scaling. Hence $\mathbb{C}[\mathcal{N}]^{\mathbb{C}^\times}=\mathbb{C} \Rightarrow \mathbb{C}[\mathcal{N}/\!\!/G_0]^{\mathbb{C}^\times}=\mathbb{C}$.

On the other hand, Proposition 2 in Sec. 1.0. and the surjectivity of $\mathcal{N}\longrightarrow \mathcal{N}/\!\!/G_0$ imply that $\mathcal{N}/\!\!/G_0$ is finite (as a set). Since $\mathbb{C}^\times$ is connected, it's action on a finite variety is trivial. Hence $\mathbb{C}[\mathcal{N}/\!\!/G_0]^{\mathbb{C}^\times}=\mathbb{C}[\mathcal{N}/\!\!/G_0]$.

Case 2: x is general. Set $L=Z_G(x_s)^\circ$ (in fact, $Z_G(x_s)$ is already connected – this is a so called Levi subgroup of G) Adapting an argument of the proof of Proposition in Sec 1.2 of Lec 5) we see that L is reductive. Since $\theta(x_s)=\varepsilon x_s$ ($x\in\mathfrak{g}_1 \Rightarrow x_s\in\mathfrak{g}_1$ by Corollary in Sec 2.1 of Lec 6), we have $\theta(Z_G(x_s))=$ $=Z_G(\theta(x_s))=Z_G(x_s)$. So L is θ-stable.

Note that $x_s, x_n\in\mathfrak{g}_1\cap\mathfrak{l}=\mathfrak{l}_1$ and $\overline{L_0x_n}\ni 0$ by Case 1. Also, L_0 fixes $x_s \Rightarrow L_0x=x_s+L_0x_n \Rightarrow x_s\in\overline{L_0x}\subset\overline{G_0x}$. $\square$

From the proof we can deduce property (6) from the intro to Sec 1 in Lec 7.

Corollary (of the proof): Let $\pi:\mathfrak{g}_1\twoheadrightarrow\mathfrak{g}_1/\!\!/G_0$ denote the quotient morphism. Every fiber consists of finitely many orbits.

Proof: Recall, Sec 1.4 of Lec 3, that the points of $\mathfrak{g}_1/\!/G_0$ are in bijection with the closed G_0-orbits in $\mathfrak{g}_1$. By Propositions 1 & 3 those are exactly the semisimple orbits. Moreover, Proposition 3 shows that $x,y\in\mathfrak{g}_1$ are in the same fiber of $\pi \Leftrightarrow G_0x_s = G_0y_s$.

Exercise: $\forall\, x\in\mathfrak{g}_1$ there's a bijection

$$\begin{array}{ccc} \{\text{nilpotent } Z_G(x_s)^\theta\text{-orbits in } \mathfrak{l}_1\} & \xrightarrow{\sim} & \{G_0\text{-orbits } G_0y \text{ w. } G_0y_s = G_0x_s\} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ Z_G(x_s)^\theta z & \longmapsto & G_0(x_s+z) \end{array}$$

The set in the l.h.s. is finite by Proposition 2. $\square$

1.2) Proof of Theorem

Geometrically, we have the unique morphism $\underline{c}: \mathfrak{a}/W_\theta \to \mathfrak{g}_0/\!/G_1$ making the following diagram commutative

$$\begin{tikzcd} \mathfrak{a} \arrow[r, hook, "c"] \arrow[d] & \mathfrak{g}_1 \arrow[d] \\ \mathfrak{a}/W_\theta \arrow[r, "\underline{c}"'] & \mathfrak{g}_1/\!/G_0 \end{tikzcd}$$

and we want to prove that $\underline{c}$ is an isomorphism. Since $\mathfrak{g}_1/\!/G_0$ is normal it's enough to show $\underline{c}$ is bijective (cf. Sec 1 of Lec 4)

Step 1 (surjectivity)

Let $\chi\in\mathfrak{g}_1/\!/G_0$ and $x\in\pi^{-1}(\chi)$ lie in the unique closed G_0-

orbit. The element x is semisimple by Prop 3 hence lies in some Cartan subspace ($\mathbb{C}x$ is a subspace consisting of pairwise commuting s/simple elements). We can replace x w. a conjugate & assume $x \in \mathfrak{a}$. Then $\underline{c}(W_\theta x) = X$, so $\underline{c}$ is surjective.

Step 2 (injectivity)

Since $G_0 x$ is closed $\forall x \in \mathfrak{a}$ and each fiber of π contains a unique closed G_0-orbit the injectivity reduces to checking:

(*) $\quad x \in G_0 y$ for $x, y \in \mathfrak{a} \Rightarrow x \in N_{G_0}(\mathfrak{a}) y$

So suppose $G_0 x = G_0 y \Leftrightarrow \exists\, g \in G_0$ w. $g.y = x$. Both Cartan subspaces $\mathfrak{a}$ and $g.\mathfrak{a}$ contain x. Let $L = Z_G(x)^\circ$. Then $\mathfrak{a}, g.\mathfrak{a} \subset \mathfrak{l}_1$ and are Cartan subspaces there. Hence $\exists\, h \in L_0$ w. $g.\mathfrak{a} = h.\mathfrak{a}$. Note that $h.x = x$. Hence $(h^{-1}g).y = x$ & $h^{-1}g.\mathfrak{a} = \mathfrak{a} \Leftrightarrow h^{-1}g \in N_{G_0}(\mathfrak{a})$ finishing the proof. $\square$

2) Weyl group is a complex reflection group.

2.1) Complex reflection groups.

Definition: Let V be a finite dimensional vector space$/\mathbb{C}$.

- $s \in GL(V)$ is called a complex reflection (a.k.a. pseudo-reflection) if it has finite order & $\mathrm{rk}(s - id) = 1$
- A finite subgroup $W \subset GL(V)$ is called a complex reflection

subgroup if it is generated by complex reflections.

Examples: 1) A complex reflection group preserving a real form $V_{\mathbb{R}} \subset V$ (resp. a rational form $V_{\mathbb{Q}} \subset V$) is the same thing as a real reflection group (resp. a crystallographic reflection group a.k.a. the Weyl group of a semisimple Lie algebra).

2) If $\dim V = 1$, then any finite subgroup of $GL(V)$ is a complex reflection group.

Here is the main reason why one cares about complex reflection groups is the following result.

Thm (Chevalley-Shephard-Todd). For a finite subgroup $W \subset GL(V)$ TFAE:

1) W is a complex reflection group
2) $V/\!/W$ is an affine space
3) $\pi: V \twoheadrightarrow V/\!/W$ is flat ($\Leftrightarrow \mathbb{C}[V]$ is a free $\mathbb{C}[V]^W$-module).

We won't give a complete proof, it can be found in [B], Ch. V, §5. Some implications are easier. For example, 2) $\Rightarrow$ 3) follows from a basic commutative algebra observation that a finite dominant morphism $\mathbb{A}^n \to \mathbb{A}^n$ is flat. We will prove 2) $\Rightarrow$ 1) below.

2.2) Main result

Thm 1 (Vinberg): $W_\theta \subset GL(\mathfrak{a})$ is a complex reflection group.

Combining this with theorems from Secs 1.0 and 2.1 we arrive at:

Corollary: $\mathfrak{g}_1 /\!/ G_0 \xrightarrow{\sim} \mathfrak{a} /\!/ W_\theta$ is an affine space.

Vinberg's proof in the general situation is not pleasant (involves some case by case considerations). We will only prove an important special case: when G_0 is semisimple. Here we have the following cute general result due to Panyushev.

Theorem 2: Let U, V be finite dimensional $\mathbb{C}$-vector spaces, and $\Gamma \subset GL(U)$ & $G \subset GL(V)$ be finite and (connected) s/simple subgroups, respectively. If $U/\!/\Gamma$ & $V/\!/G$ are isomorphic as varieties, then Γ is a complex reflection group (and hence $U/\!/\Gamma$ is an affine space).

The proof is essentially topological & uses the following concept.

Definition: Let X be an irreducible variety over $\mathbb{C}$. We say that X is strongly simply connected if $X \setminus Y$ is simply connected $\forall$ closed subvariety $Y \subset X$ w. $\operatorname{codim}_X Y \geq 2$.

Example/exercise: $\mathbb{A}^n$ is strongly simply connected.

Panyushev's theorem follows from the following two results to be proved next time. We use the notation of Thm 2 for both.

Proposition 1: $V/\!\!/G$ is strongly simply connected.

Proposition 2: If $U/\!\!/\Gamma$ is strongly simply connected, then Γ is a complex reflection group.