Invariant theory 8, 02/05/25

1) Comparison between invariants of s/simple & finite groups.
2) Computation of G_0 & $\mathfrak{g}_1$

Refs: [PV], Sec 8.3; [OV], Sec 4.4

1.0) Reminder

In Sec 2.2 of Lec 7 we have stated the following theorem due to Panyushev:

Theorem: Let U, V be finite dimensional $\mathbb{C}$-vector spaces, and $\Gamma \subset GL(U)$ & $G \subset GL(V)$ be finite and (connected) s/simple subgroups, respectively. If $U/\!/\Gamma$ & $V/\!/G$ are isomorphic as varieties, then Γ is a complex reflection group.

To prove this we introduced the following definition & stated two propositions to be proved in this lecture.

Definition: Let X be an irreducible variety over $\mathbb{C}$. We say that X is strongly simply connected if $X \setminus Y$ is simply connected $\forall$ closed subvariety $Y \subset X$ w. $\operatorname{codim}_X Y \geqslant 2$.

Proposition 1: $V/\!/G$ is strongly simply connected.

Proposition 2: If $U/\!/\Gamma$ is strongly simply connected, then Γ is a complex reflection group.

1.1) Proof of Proposition 1

We will show that for any G-stable divisor $D \subset V$, we have $\operatorname{codim}_{V/\!/G} \pi(D) = 1$. We will use this to prove Proposition. We can assume D is irreducible.

Step 1: Let $f \in \mathbb{C}[V]$ be s.t. $D = f^{-1}(0)$, it's defined uniquely up to multiplication w. an invertible function, i.e. a nonzero scalar. We claim that $f \in \mathbb{C}[V]^G$.

Note that since D is G-stable, $D = [g.f]^{-1}(0)$ $\forall g \in G$. So $\mathbb{C}f \subset \mathbb{C}[V]$ is G-stable. Since $\mathbb{C}[V]$ is a rational representation of G, so is $\mathbb{C}f$. Thus the representation of G in $\mathbb{C}f$ gives rise to an algebraic group homomorphism $G \to \mathbb{C}^\times$. Since G is connected & s/simple, such a homomorphism is trivial proving $f \in \mathbb{C}[V]^G$.

Step 2: We have the short exact sequence of G-modules:

$$0 \longrightarrow \mathbb{C}[V] \xrightarrow{f} \mathbb{C}[V] \longrightarrow \mathbb{C}[D] \longrightarrow 0.$$

Thx to the complete reducibility it remains exact after taking G-invariants leading to

$$\mathbb{C}[D]^G \xrightarrow{\sim} \mathbb{C}[V]^G/\mathbb{C}[V]^G f \qquad (*)$$

Since $\pi(D) \simeq D/\!/G$ (Sec 1.4 in Lec 3), $(*)$ implies $\operatorname{codim}_{V/\!/G} \pi(D) = 1$ proving the claim.

Step 3: We will need the following basic facts on the topology of algebraic varieties & morphisms.

Fact 1 (harder): any irreducible algebraic variety$/\mathbb{C}$ is connected in the usual topology.

Fact 2 (easier): let X be a smooth irreducible variety & $Y \subset X$ a closed subvariety. Then:

(a) If $X \setminus Y$ is simply connected, then X is so.

(b) Assume $\operatorname{codim}_X Y \geqslant 2$. If X is simply connected, then so is $X \setminus Y$.

Fact 3 (hard) Let $\varphi: X \to Y$ be a morphism of smooth varieties. Then $\exists$ open dense subvariety $Y^\circ \subset Y$ s.t. $\varphi: \pi^{-1}(Y^\circ) \to Y^\circ$ is a locally trivial fibration in the usual topology.

The latter follows from [Ver].

Step 4: We claim that every fiber of π is connected in the usual topology. Indeed, every orbit closure is irreducible, so connected by Fact 1. Also for x, y w. $\pi(x) = \pi(y)$, $\overline{Gx}$ & $\overline{Gy}$ both contain the unique closed orbit in $\pi^{-1}(\pi(x))$ that is also connected. To deduce that $\pi^{-1}(\pi(x))$ is connected is an exercise.

Step 5: Here we Steps 2, 4 & facts from Step 3 to finish the proof of the claim that $X := V/\!/G$ is strongly simply connected. We write X^{reg} for the locus of smooth points.

Exercise: X is strongly simply connected $\Leftrightarrow$ X^{reg} is simply connected (hint: X is normal $\Rightarrow$ $\operatorname{codim}_X (X \setminus X^{reg}) \geq 2$)

Let $V^\circ := \pi^{-1}(X^{reg})$. Thx to Step 2, $V \setminus V^\circ$ cannot contain a divisor, so $\operatorname{codim}_V V \setminus V^\circ \geq 2$. By Fact 2, V° is simply connected. By Fact 3, $\exists$ open dense $U \subset X^{reg}$ s.t. $\pi : \pi^{-1}(U) \longrightarrow U$ is a locally trivial fibration. By Step 4, the fibers are connected.

Now take $x \in U$ & let γ be a loop in X^{reg} w. $\gamma(0) = x$. We can deform γ so that it's contained in U. Then we can lift it to a loop, $\tilde{\gamma}$, in $\pi^{-1}(U)$ b/c the fibers are connected.

Since V° is simply connected, we can find a homotopy $\kappa: [0,1]^2 \longrightarrow V^\circ$ connecting $\tilde{\gamma}$ with the trivial. Then $\pi \circ \kappa$ is a homotopy connecting γ with the trivial loop hence finishing the proof. $\square$

1.2) Proof of Proposition 2.

Let Γ' be the subgroup of Γ generated by the complex reflections, it's normal. We need to show that $\Gamma' = \Gamma$.

Consider the action of $H := \Gamma/\Gamma'$ on $X := U/\!/\Gamma'$. For $h \in H$ we write X^h for the fixed points of h.

Lemma: If $h \neq 1$, then $\operatorname{codim}_X X^h \geqslant 2$.

Proof: Recall that the points of X are in bijection with the Γ'-orbits in U (via taking the fiber of the quotient morphism). We have $h(\Gamma' u) = \Gamma' u \Leftrightarrow \exists\, \gamma \in \Gamma$ w. $\gamma\Gamma' = h$ & $\gamma u = u$. So $X^h = \bigcup_{\gamma | \gamma\Gamma' = h} \pi'(U^\gamma)$, where $\pi': U \to X$ is the quotient morphism. Note that $\Gamma \setminus \Gamma'$ contains no complex reflections. So $\operatorname{codim}_U U^\gamma \geqslant 2$. Now the lemma follows from the next exercise. $\square$

Exercise 1: Let H be a finite group acting on an affine variety X. Then $\mathbb{C}[X]$ is integral over $\mathbb{C}[X]^H$, hence the quotient morphism $\pi: X \to X/\!/H$ is finite.

We keep the notation of the exercise. Let X° be an open H-stable subvariety. Then $Y := X \setminus X^\circ$ is H-stable & closed. So $\pi(Y) \subset X/\!/H$ is closed & parameterizes the H-orbits in Y. It follows that $(X/\!/H)^\circ := (X/\!/H) \setminus \pi(Y)$ parameterizes the H-orbits in X° meaning that each fiber of $\pi: X^\circ \to (X/\!/H)^\circ$ is a single orbit. By Exercise 1, this morphism is finite.

Exercise 2: If H acts on X° freely, then $\pi: X^\circ \to (X/\!/H)^\circ$ is etale. Hint: for $\chi \in X/\!/H$, let $\mathbb{C}[X/\!/H]^{\wedge_\chi}$ denote the completion of $\mathbb{C}[X/\!/H]$ at the maximal ideal of χ. Let $\mathbb{C}[X]^{\wedge_{\pi^{-1}(\chi)}}$ be the completion at the vanishing ideal of $\pi^{-1}(\chi)$. Establish an H-equivariant isomorphism $\mathbb{C}[X]^{\wedge_{\pi^{-1}(\chi)}} \xrightarrow{\sim} \mathbb{C}[X/\!/H]^{\wedge_\chi} \otimes_{\mathbb{C}[X/\!/H]} \mathbb{C}[X]$.

Proof of Proposition 2: Take $X = U/\!/\Gamma'$, $H = \Gamma/\Gamma'$ and let X° be the subset of all points in X^{reg} with trivial stabilizer in H. Since X is normal, we have $\operatorname{codim}_X (X \setminus X^{reg}) \geq 2$, and by Lemma $\operatorname{codim}_{X^{reg}} (X^{reg} \setminus X^\circ) \geq 2 \Rightarrow \operatorname{codim}_X (X \setminus X^\circ) \geq 2$. Exercise 2 implies that $X^\circ \to (X/\!/H)^\circ$ is finite & etale cover. Hence, if $H \neq \{1\}$,

$(X/\!/H)^\circ$ has a nontrivial topological cover, X° & hence is not simply connected. From Exercise 1 we deduce that

$$\operatorname{codim}_{X/\!/H}\left((X/\!/H)\setminus(X/\!/H)^\circ\right)\geqslant 2$$

Hence $X/\!/H$ is not strongly simply connected, a contradiction. $\square$

2) Computation of G_0 & $\mathfrak{g}_1$

Our goal in this section (and the next lecture) is to explain how one can produce examples of $G_0 \curvearrowright \mathfrak{g}_1$. Let $\mathfrak{g}$ be simple.

2.1) Case of inner θ.

Let G_{ad}, G_{ss} denote the adjoint & simply connected groups with Lie algebra $\mathfrak{g}$ (where $\mathfrak{g}$ is a simple Lie algebra). Then $G_{ad}=\operatorname{Aut}(\mathfrak{g})^\circ$ & $G_{ss}\twoheadrightarrow G_{ad}$ with kernel naturally identified with $P^\vee/Q^\vee$, where $P^\vee\supset Q^\vee$ are the coweight & coroot lattices of $\mathfrak{g}$. Let $T\subset G_{ss}$ be a maximal torus (= a maximal w.r.t. $\subseteq$ subgroup of G_{ss} which is a torus), equivalently a connected subgroup of G whose Lie algebra is a Cartan, to be denoted by $\mathfrak{h}$. We start by considering the easier situation: $\theta\in G_{ad}$, in the next lecture we will consider the general situation. Let $\tilde{\theta}$ be a preimage of θ in G_{ss}. We make the following assumption. Later, we will comment on its status.

Assumption: $\tilde{\theta} \in T$.

Consider the map $\mathfrak{h} \to T$, $x \mapsto \exp(2\pi\sqrt{-1}\,x)$. This is an abstract group epimorphism (b/c $T \simeq (\mathbb{C}^\times)^n$) whose kernel is $Q^\vee$. Let ν lie in the preimage of $\tilde{\theta}$.

Let's analyze the condition that the order of θ divides d (we discuss the equality later). $\theta = \mathrm{Ad}\,\tilde{\theta}$ acts on $\mathfrak{h}$ by 0 and on the root space $\mathfrak{g}_\beta$, it acts by $\exp(2\pi\sqrt{-1}\langle\beta,\nu\rangle)$.

So TFAE:

- the order of θ divides d
- $\langle\beta,\nu\rangle \in \frac{1}{d}\mathbb{Z}$ $\forall$ roots $\beta \Leftrightarrow \nu \in \frac{1}{d}P^\vee$.

Note that applying an element of W to $\tilde{\theta}$ (and ν) leads to a conjugate automorphism (hence essentially the same G_0 & $\mathfrak{g}_1$), while adding an element of $Q^\vee$ to ν doesn't change $\tilde{\theta}$. So we need to describe the orbit set $(\frac{1}{d}P^\vee)/(W \ltimes Q^\vee)$.

An important observation is that $W \ltimes Q^\vee$ viewed as a group of affine transformations of $\mathbb{R}\otimes_{\mathbb{Z}} P^\vee$ (a real form of $\mathfrak{h}$) is generated by affine reflections (w.r.t. the hyperplanes $\beta = n$ for a root β & $n \in \mathbb{Z}$). As such it has a fundamental domain: the polytope given as follows: let $\beta_1, \dots \beta_r$ be the simple roots of $\mathfrak{g}$ & β_0 be the maximal root. Then the "alcove"

$$A = \{x \in \mathbb{R} \otimes_{\mathbb{Z}} P^\vee \mid \langle \beta_i, x \rangle \geq 0 \ \forall i = 1, \dots r \ \& \ \langle \beta_0, x \rangle \leq 1\}$$

is a fundamental domain. Of course, $x \in \frac{1}{d} P^\vee \Leftrightarrow \langle \beta_i, x \rangle \in \frac{1}{d} \mathbb{Z}, \forall i.$

Exercise 1: TFAE:

- The order of θ is exactly d.
- the vector $d \left(\langle \beta_i, \nu \rangle \right)_{i=0}^{r} \in \mathbb{Z}^{r+1}$ is primitive