# Invariant theory 9, 2/10/24

1) Computation of $(G_0, \mathfrak{g}_1)$
2) Example of $SL_3 \curvearrowright S^3(\mathbb{C}^3)$, started
Refs: [OV], Sec. 4.4; [K], Ch. 6-8.

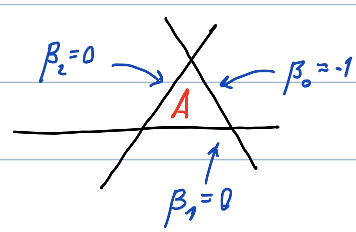
## 1) Computation of $(G_0, \mathfrak{g}_1)$

### 1.1) Inner case: examples of automorphisms.

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$, $G_{sc} \twoheadrightarrow G_{ad} = Ad(\mathfrak{g})^\circ$ be simply connected & adjoint groups w. Lie algebra $\mathfrak{g}$. We considered the situation when $\theta$ is an order $d$ element of $G_{ad}$. We chose a lift $\tilde{\theta} \in G_{sc}$ and assumed that $\tilde{\theta}$ is in a maximal torus $T \subset \tilde{G}$. Let $P^\vee$ denote the coweight lattice in $\mathfrak{h} := Lie(T)$. Let $\beta_1, \dots \beta_r \in \mathfrak{h}^*$ be a system of simple roots & $\beta_0$ be the <u>minimal</u> root ($=(-1)\cdot$ maximal root - notice the change of notation from Lecture 8). Consider the fundamental alcove

$$A = \{x \in \mathbb{R} \otimes_{\mathbb{Z}} P^\vee \mid \langle \beta_i, x\rangle \geqslant 0, \langle \beta_0, x \rangle \geqslant -1\}$$

**Example**: $\mathfrak{g} = \mathfrak{sl}_3$. Then $\beta_0 = \beta_1 + \beta_2$ & $A$ is as follows:




Last time we explained that we can reduce the case of general

$\tilde{\theta} \in T$ w. $Ad(\tilde{\theta})$ of order $d$ to $\tilde{\theta} = \exp(2\pi\sqrt{-1}\,\lambda)$ w. $\lambda \in A \cap \frac{1}{d}P^\vee$

Two questions arise:

- How to describe $\lambda$ more combinatorially &
- How to recover $G_0$ & $\mathfrak{g}_1$ from this description?

Here is a combinatorial way to encode an element of $A \cap \frac{1}{d}P^\vee$. Consider the extended (by the root $\beta_0$) Dynkin diagram of $\mathfrak{g}$ (an "untwisted" affine Dynkin diagram). Let $(a_0 = 1, a_1, \dots a_r)$ be unique positive integers s.t. $\sum_{i=0}^{r} a_i \beta_i = 0$

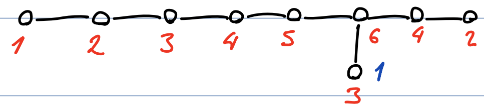
Exercise: There is a bijection between the points of $A \cap \frac{1}{d}P^\vee$ and tuples $(n_0, \dots n_r) \in \mathbb{Z}_{\geq 0}^{r+1}$ w. $\sum_{i=0}^{r} a_i n_i = d$ (sending $\lambda$ to

$$d(1 + \langle \beta_0, \lambda \rangle, \langle \beta_1, \lambda \rangle, \dots, \langle \beta_r, \lambda \rangle))$$

Moreover $\lambda$ corresponds to an order $d$ automorphism $\Leftrightarrow$ the vector $(n_0, \dots, n_r) \in \mathbb{Z}^{r+1}$ is primitive.

The tuples $(n_0, \dots, n_r)$ are convenient to depict on the extended affine Dynkin diagram (sometimes, this is referred to as a Kac diagram).

Example: Here's a diagram giving an order 3 automorphism of $E_8$ (we mark $a_i$'s in red and $n_i$'s in blue – we only mark non-zero entries)




Remark: One can talk about semisimple elements in algebraic groups similarly to their Lie algebras; finite order implies semisimple. And every semisimple element is contained in some maximal torus. From here one can establish a bijection between

- the $G_{sc}$-conjugacy classes of elements $\tilde{\theta} \in G_{sc}$ s.t. $Ad(\tilde{\theta})^d = id$
- and points of $A \cap \frac{1}{d} P^\vee$.

## 1.2) Inner case: computation of $G_0$ & $\mathfrak{g}_1$

Here we explain how to fully compute $G_0$ & partly compute $\mathfrak{g}_1$ starting from the Kac diagram. Let $\omega_1^\vee, \dots \omega_r^\vee$ denote the fundamental coweights so that $\nu = \sum_{i=1}^{r} \frac{n_i}{d} \omega_i^\vee$.

Exercise 1: For $\tilde{\theta} = \exp(2\pi\sqrt{-1}\,\nu)$, the element $Ad(\tilde{\theta})$ acts on $\mathfrak{h}$ by 1 & on a root space $\mathfrak{g}_\beta$ by $\exp(2\pi\sqrt{-1}\langle \nu, \beta\rangle)$

So $\mathfrak{g}_0$ is the direct sum of $\mathfrak{h}$ & all $\mathfrak{g}_\beta$ s.t.

(1) $$d\langle \nu,\beta\rangle = \sum_{i=1}^{r} n_i \langle \omega_i^\vee, \beta\rangle \,\vdots\, d$$

For $\mathfrak{g}_1$ we have the direct sum of all $\mathfrak{g}_\beta$ w. $d\langle\nu,\beta\rangle-1$ divisible by 1.

One can give a more explicit description of $\mathfrak{g}_0$.

Lemma: $\mathfrak{g}_0$ is the direct sum of:

- The annihilator of $\{\beta_i \mid n_i=0\}$ in $\mathfrak{h}$
- The semisimple algebra whose simple roots are $\beta_i$ w. $n_i=0$, i.e. the Dynkin diagram consists of the nodes of Kac's diagram labelled 0.

Proof: Set $\mathfrak{l} := \mathfrak{h} \oplus \bigoplus_{\beta \mid \langle\beta,\nu\rangle=0} \mathfrak{g}_\beta$. Since $\nu$ is dominant, the roots $\beta_i$ w. $i>0$ & $n_i=0$ are a system of simple roots for $[\mathfrak{l},\mathfrak{l}]$. The following claims imply the lemma (b/c $\mathfrak{g}_0 \supset \mathfrak{h}$):

(a) if $n_0 \neq 0$, then $\mathfrak{g}_0 = \mathfrak{l}$

(b) if $n_0=0$, then $\mathfrak{g}_0 = \mathfrak{l} \oplus U(\mathfrak{l})\mathfrak{g}_{\beta_0} \oplus U(\mathfrak{l})\mathfrak{g}_{-\beta_0}$, where $U(\mathfrak{l})$ is the universal enveloping algebra acting on $\mathfrak{g}$ via $\mathrm{ad}|_{\mathfrak{l}}$.

Since $\nu$ is dominant $\Rightarrow \langle\nu,\beta_0\rangle \leq \langle\nu,\beta\rangle \leq \langle\nu,-\beta_0\rangle$ $\forall$ roots $\beta$. On the other hand, $\langle\nu,-\beta_0\rangle = 1-\frac{n_0}{d}$. So if $n_0 \neq 0$, then $\langle\nu,\beta\rangle \in \mathbb{Z}$ $\Leftrightarrow \langle\nu,\beta\rangle=0$ and (a) follows.

Assume $n_0=0 \Leftrightarrow \langle\nu,\beta_0\rangle=-1$. Then $\langle\nu,\beta\rangle\in\mathbb{Z} \Leftrightarrow \langle\nu,\beta\rangle\in\{-1,0,1\}$.

Assume $\langle\nu,\beta\rangle=1$. Then $\beta_0+\beta = \sum_{i=1}^{r} m_i\beta_i$ w. $m_i \leq 0$. Since

$\langle \nu, \beta+\beta_0 \rangle = 0$ & $\nu$ is dominant, we get $m_i = 0$ if $n_i > 0$. Note that $\mathfrak{g}$ is an irreducible $\mathfrak{g}$-module (via ad) & $-\beta_0$ is a highest weight. From $\beta = -\beta_0 + \sum_{i \mid n_i = 0} m_i \beta_i$ w. $m_i \leq 0$ we deduce $\mathfrak{g}_\beta \subset \mathcal{U}(\mathfrak{l})\mathfrak{g}_{\beta_0}$.

The case $\langle \nu, \beta \rangle = -1$ is handled similarly (exercise). $\square$

To determine $\mathfrak{g}_1$ as a $G_0$-representation one can use the following observation:

Exercise 2: Let $n_i > 0$. Then $\beta_i$ is an anti-dominant weight for $\mathfrak{g}_0$, & and $\mathfrak{g}_{\beta_i}$ is annihilated by $\mathfrak{n}_- \subset \mathfrak{g}_0$ (the maximal nilpotent subalgebra corresponding to negative roots, so $\mathcal{U}(\mathfrak{g}_0)\mathfrak{g}_{\beta_i}$ is the irreducible $\mathfrak{g}_0$-module $\underline{V}(\beta_i)$ with lowest weight $\beta_i$.

In some cases this exercise and an easy dimension count suffice to fully determine $\mathfrak{g}_1$ (one can determine $\mathfrak{g}_1$ in the general case but we won't need this).

Example: We return to the $E_8$ example from before. Lemma shows that $\mathfrak{g}_0$ is of type $A_8$, i.e. $\mathfrak{g}_0 \simeq \mathfrak{sl}_9$. As a weight of $\mathfrak{sl}_9$ $\beta_i$ w. $n_i = 1$ is $-\omega_6$ (or $-\omega_3$ depending on the numbering). So $\underline{V}(-\omega_6) = \Lambda^3 \mathbb{C}^9$ is a direct summand in $\mathfrak{g}_1$ by Exercise 2. Since the Killing form on $\mathfrak{g}$ restricts to a non-degenerate pairing $\mathfrak{g}_i \times \mathfrak{g}_{-i} \to \mathbb{C}$, we have $\mathfrak{g}_{-1} \simeq_{G_0} \mathfrak{g}_1^*$. Then we note that

$$\dim \mathfrak{g}_0 + \dim V(\omega_3) + \dim V(\omega_6) = 80 + 84 + 84 = 248 = \dim \mathfrak{g}$$

$$\Rightarrow \mathfrak{g}_1 = \Lambda^3 \mathbb{C}^9.$$

In particular we see that the action of $G = SL_9$ on $V = \Lambda^3\mathbb{C}^9$ has nice invariant theoretic properties. It turns out that the degrees of free homogeneous generators are 12, 18, 24, 30, in particular that the Weyl group is not a real reflection group (bonus exercise: explain why).

### 1.3) General case (of $\theta$)

We continue to assume $\mathfrak{g}$ is simple. Let $T \subset B$ be a maximal torus & Borel subgroups in $G_{sc}$, they determine a system of simple roots in $\mathfrak{h}^*$. Let $\Pi$ denote the Dynkin diagram of $\mathfrak{g}$ & $\mathrm{Aut}(\Pi)$ denote the automorphisms of $\Pi$ ($S_3$ for $D_4$; $S_2$ for $A_n$, $D_n$ w. $n \neq 4$, $E_6$; trivial else). Then $\mathrm{Aut}(\Pi)$ can be viewed as the group of automorphisms of $G_{sc}$ preserving $B$ & $T$. We have

$$\mathrm{Aut}(\mathfrak{g}) = \mathrm{Aut}(\Pi) \ltimes G_{ad}.$$

Pick $\tau \in \mathrm{Aut}(\Pi)$, and let $e \in \{1,2,3\}$ denote its order. Let $\mathfrak{h}_\tau := \{x \in \mathfrak{h} \mid \tau(x) = x\}$. We are looking for automorphisms of the form $\theta = \mathrm{Ad}(\tilde{\theta})$ for $\tilde{\theta} = \tau \cdot \exp(2\pi\sqrt{-1}\,\lambda)$, where $\lambda$ is a rational point of $\mathfrak{h}_\tau$. Note that $\tau$ & $\exp(2\pi\sqrt{-1}\,\lambda)$ commute so $\theta$ has finite order. Similarly to the inner case, one can show that any finite order element of

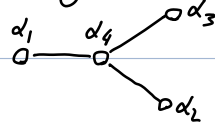
$\{\tau\}\times G_{sc}\subset \mathrm{Aut}(\Pi)\ltimes G_{sc}$ is conjugate to one of this form.

In the inner case ($\tau=1$) we then reduced to the case when $\nu$ lies in a fundamental alcove for the affine Weyl group. Something similar happens in the general case – but the affine root system is twisted for $\tau\neq 1$.

In the general case the affine root system we need is a subset of $\mathfrak{h}_\tau^*\times\mathbb{Z}$ consisting of all elements $(\alpha,n)\neq(0,0)$ s.t. $\alpha$ is a weight of $\mathfrak{h}_\tau$ in $\mathfrak{g}_{[n]}:=\ker(\tau-\epsilon^n)$ ($\epsilon=\exp(2\pi\sqrt{-1}/e)$, $e=\mathrm{ord}(\tau)$).

We are not going to prove this is an affine root system – see [OV], Sec 4.4 for this (and [K], Secs 6-8 for affine root systems) – but will give an example.

*Example*: We consider $\mathfrak{g}=\mathfrak{so}_8$ and its order 3 diagrams automorphism $\tau$. We label the simple roots as follows




Then $\tau$ fixes $\alpha_4$ and permutes $\alpha_1,\alpha_2,\alpha_3$ cyclically.

The $\tau$-orbits in the set of positive roots are as follows:

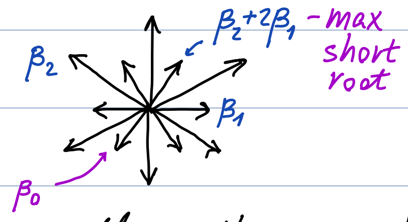
(1) $\alpha_4$, $\sum_{i=1}^4\alpha_i$ & $2\alpha_4+\sum_{i=1}^3\alpha_i$ are singletons

(2) $\alpha_j$, $j\in\{1,2,3\}$

(3) $\alpha_4+\alpha_j$, $j\in\{1,2,3\}$ ← permuted cyclically.

(4) $(\sum_{i=1}^4\alpha_i)-\alpha_j$, $j\in\{1,2,3\}$

A basis in $\mathfrak{h}_\tau$ is formed by $h_1:=\alpha_0^\vee$ & $h_2:=\alpha_1^\vee+\alpha_2^\vee+\alpha_3^\vee$. The weights of $\mathfrak{g}_{[0]}$ are 0 & the restrictions of (1)-(4) (and their negatives) to $\mathfrak{h}_\tau$. They form a root system of type $G_2$ with (1) giving the long positive roots & (1)-(3) giving the short positive roots (w. short simple root, $\beta_1$, being the restriction of $\alpha_i$, $i=1,2,3$, and the long simple root, $\beta_2$, being the restriction of $\alpha_4$).




Using this one shows that $\mathfrak{g}_{[0]}$ is $G_2$.

The weights in $\mathfrak{g}_{[\pm 1]}$ are 0 (for $\mathfrak{g}_{[\pm 1]}\cap\mathfrak{h}$) and the restrictions of (1)-(3), all w. multiplicity 1 (for example, the weight space corresponding to $\beta_1$ is the $\epsilon$-eigenspace of $\tau$ in $\mathfrak{g}_{\alpha_1}\oplus\mathfrak{g}_{\alpha_2}\oplus\mathfrak{g}_{\alpha_3}$). From here we see that both $\mathfrak{g}_{[\pm 1]}$ are isomorphic to the 7-dimensional irreducible representation of $G_2$.

The resulting affine root system is known as $D_4^{(3)}$ & is Dynkin diagram is $\underset{1}{\overset{\beta_0}{\circ}}-\underset{2}{\overset{\beta_1}{\circ}}\Lleftarrow\underset{1}{\overset{\beta_2}{\circ}}$ (we adjoin the antidominant short root $\beta_0$ - corresponding to the lowest weight of $\mathfrak{g}_{[1]}$). Then $\beta_0+2\beta_1+\beta_2=0$.

In the remaining cases (where $\tau$ has order 2) we see the same picture: we take the Dynkin diagram of $\mathfrak{g}_{[0]}$ and adjoin the lowest weight of $\mathfrak{g}_{[1]}$.

The classification of finite order automorphisms in terms of the Kac diagrams proceeds like in the inner case but the procedure is more tricky. We will only consider an example

Example (cont'd) Here we consider $\theta$ corresponding to the Kac diagram $\underset{1}{\circ}-\underset{2}{\circ}\Lleftarrow\overset{1}{\underset{1}{\circ}}$ ($\tau$ itself corresponds to $\overset{1}{\underset{1}{\circ}}-\underset{2}{\circ}\Lleftarrow\underset{1}{\circ}$). The Kac diagram just means that $\theta$ acts of $\mathfrak{g}_{[0],\beta_1}$ & $\mathfrak{g}_{[1],\beta_0}$ trivially and on $\mathfrak{g}_{[0],\beta_2}$ by $\epsilon$. Then $\tilde{\theta}=\tau\exp(2\pi\sqrt{-1}\,\nu)$, where $\langle\beta_1,\nu\rangle=0$, $\langle\beta_2,\nu\rangle=\frac{1}{3}$ (this $\nu$ is uniquely defined b/c $\beta_1,\beta_2$ form a basis in $\mathfrak{h}_\tau^{*}$). The fixed point subalgebra $\mathfrak{g}_0=\mathfrak{g}^\theta$ is 8-dimensional, it's the direct sum of $\mathfrak{h}_\tau$ & the 1-dimensional wt. spaces $\mathfrak{g}_{[0],\pm\beta_1}, \mathfrak{g}_{[\pm1],\pm\beta_0}, \mathfrak{g}_{[\pm1],\pm(\beta_0+\beta_1)}$. As one can guess from the Kac diagram (and, in fact, prove), $\mathfrak{g}_0\simeq\mathfrak{sl}_3$. The $\mathfrak{g}_0$-subrepresentation $U(\mathfrak{g}_0)\mathfrak{g}_{\beta_2}\subset\mathfrak{g}_1$ is irreducible w. lowest weight $\beta_2$, which is $-3\omega_2$ (or $-3\omega_1$ depending on the numbering of simple roots for $\mathfrak{g}_0$; 3 here is $-\langle\beta_2,\beta_1^\vee\rangle$ in the $G_2$ root system). The dimension count similar to Sec 1.2 shows $\mathfrak{g}_1\simeq V(3\omega_1)$ ($=S^3(\mathbb{C}^3)$).

## 2) Example of $SL_3 \curvearrowright S^3(\mathbb{C}^3)$

In the preceding example we have seen that this arises as a $\theta$-group. We want to compute a Cartan space $\mathfrak{a}$ (in detail in this lecture) & $W_\theta$ (a sketch in the next lecture).

**Lemma:** $\dim \mathfrak{a} = 2$

Proof: We first show $\dim \mathfrak{a} \geqslant 2$.

We can choose $\mathfrak{a}$ containing $\mathfrak{g}_1 \cap \mathfrak{h} = \mathfrak{g}_{[1]} \cap \mathfrak{h}$ spanned by $x = \alpha_1^\vee + \varepsilon \alpha_2^\vee + \varepsilon^2 \alpha_3^\vee$. The roots of $\mathfrak{g}$ that vanish on $x$ are exactly roots in (1) and their opposites. So the semisimple part, $\mathfrak{m}$, of $\mathfrak{z}_{\mathfrak{g}}(x)$ is isomorphic to $\mathfrak{sl}_3$. The corresponding root spaces lie in $\mathfrak{g}_{[0]}$ by the 1st part of Example in Sec 1.3 (the root spaces for the long roots $\pm\beta_2, \pm(\beta_2+3\beta_1), \pm(2\beta_2+3\beta_1)$) so $\theta$ acts on them by $\exp(2\pi\sqrt{-1}\,\mathrm{ad}\,\nu)$. Since $\langle \beta_1, \nu \rangle = 0$, $\langle \beta_2, \nu \rangle = \frac{1}{3}$ we see that $\mathrm{ad}(\nu)$ acts on $\mathfrak{g}_{[0],\beta_2}, \mathfrak{g}_{[0],\beta_2+3\beta_1}$ by $\frac{1}{3}$ so by $\mathrm{ad}(\mathrm{diag}(\frac{1}{3}, 0, -\frac{1}{3}))$ on $\mathfrak{m}$. In particular, $\mathfrak{m} \cap \mathfrak{g}_{-1}$ contains a s/simple element: $y = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$. So we can choose $\mathfrak{a}$ w. $x, y \in \mathfrak{a}$. Since $y \in \mathfrak{m}$, $x \notin \mathfrak{a} \Rightarrow \dim \mathfrak{a} \geqslant 2$.

Now we show that $\dim \mathfrak{a} \leqslant 2$. Note that the centralizer of $y$ in $\mathfrak{m}$ is diagonalizable (the eigenvalues are peirwise distinct). It follows that $\mathfrak{h}' := \mathfrak{z}_{\mathfrak{g}}(x,y)$ is a Cartan subalgebra in $\mathfrak{g}$. Since $x, y \in \mathfrak{g}_1$, $\mathfrak{h}'$ is $\theta$-stable. Note that $\theta$ acting on $\mathfrak{h}'$ preserves the root system. Hence $\theta|_{\mathfrak{h}'}$ is defined over $\mathbb{Q}$ and therefore the multiplicities of eigenvalues $\varepsilon$ & $\varepsilon^2$ are the same.