HW 1 Solutions

1) $G \subset \mathbb{C}^\times$, $G = \{z \mid z^n = 1\}$, $V = \mathbb{C}$, $G \curvearrowright V$ by multiplications. $\mathbb{C}[V]^G = \mathbb{C}[x^n]$, clearly works

2) a) $\mathbb{C}(V)$ is alg'c over $\mathbb{C}(V)^G$ (proved as the claim that $\mathbb{C}[V]$ is integral over $\mathbb{C}[V]^G$ is Lec 2) + $\mathbb{C}(V)$ is fin. gen'd $\Rightarrow$ $\mathbb{C}(V)$ is a finite ext'n of $\mathbb{C}(V)^G$. By Galois theory, $\mathbb{C}(V)$ is a normal extension of $\mathbb{C}(V)^G$ w. Galois group G, hence $\dim_{\mathbb{C}(V)^G} \mathbb{C}(V) = |G|$

b-c) Let $f = \frac{f_1}{f_2} \in \mathbb{C}(V)$ ($f_i \in \mathbb{C}[V]$). Replace f_2 w. $\prod_{g \in G} g.f_2$, f_1 w. $f_1 \prod_{g \neq 1} g.f_2$. Still $\frac{f_1}{f_2} = f$ but now f_2 is invariant. This shows that $\mathbb{C}(V)^{g \neq 1}$ is obtained from $\mathbb{C}[V]$ by inverting the nonzero G-invariant functions. If $f \in \mathbb{C}(V)^G$, then f_1 is also invariant. So $\mathbb{C}(V)^G = \mathrm{Frac}(\mathbb{C}[V]^G)$ & $\mathbb{C}(V) = \mathbb{C}(V)^G \otimes_{\mathbb{C}[V]^G} \mathbb{C}[V]$

3) a) First of all, note that G fixes all invertible functions in $\mathbb{C}[X]$. This is based on the following claim (to be proved later)

Claim: Let $\mathbb{C}[X]^\times$ denote the group of invertible functions. Then $\mathbb{C}[X]^\times / \mathbb{C}^\times$ is a finite rank lattice

Now G acts on $\mathbb{C}[X]^\times/\mathbb{C}^\times$. Since G is connected, and $\mathbb{C}[X]^\times/\mathbb{C}^\times$ is discrete, the action is trivial. So if $f \in \mathbb{C}[X]^\times$, then $g.f = \chi_g f$, $\chi_g \in \mathbb{C}^\times$, $\forall g \in G$. It's straightforward to see that $g \mapsto \chi_g$ is a character, so $g.f = f$ $\forall f \in \mathbb{C}[X]^\times$

Now pick $f \in \mathbb{C}[X]^G$. We have $f = \prod_{i=1}^k f_i$, where $f_1 \ldots f_k$ are irreducible elements defined uniquely up to permutation and multiplication by an invertible element. Since $g.f = \prod_{i=1}^k g.f_i$ is another decomposition, we see that there is a permutation $\sigma \in S_k$ and invertible elements $u_i \in \mathbb{C}[X]^\times$ s.t $g.f_i = u_{i,g} f_{\sigma_g(i)}$. Since G is connected, we see that $f_{\sigma_g(i)}/f_i$ is invertible. So we can assume $g.f_i = u_{i,g} f_i$. We claim that if $f \in \mathbb{C}[X]$ satisfies

$g.f = u_g f$ for $u \in \mathbb{C}[X]^\times$, then $u_g = 1$ & f is G-invariant. Indeed u_g is invariant so $u_{gh} f = (gh)f = g(hf) = g(u_h f) = u_g u_h f$ so $g \mapsto u_g$ is a group homomorphism. Since $\mathbb{C}[X]^\times / \mathbb{C}^\times$ is discrete, the image of $g \mapsto u_g$ lies in $\mathbb{C}^\times$. Since G has no characters, this image is trivial. So we see that each f_i is G-invariant. Therefore we have uniquely decomposed f into the product of irreducible elements in $\mathbb{C}[X]^G$. Hence $\mathbb{C}[X]^G$ is factorial.

b) For the same reason as in (a), every $f \in \mathbb{C}(X)^G$ decomposes as $\prod_i f_i^{d_i}$ ($d_i \in \mathbb{Z}$) for $f_i \in \mathbb{C}[X]^G$. In particular, $\mathbb{C}(X)^G = \operatorname{Frac} \mathbb{C}[X]^G$. By the Rosenlicht theorem, we can find a G-stable open $X' \subset X$ and $F_1, \dots, F_k \in \mathbb{C}[X']^G$ separating orbits in X'. We have $F_i = \frac{f_{i1}}{f_{i2}}$ for $f_{ij} \in \mathbb{C}[X]^G$. Then f_{i1}, f_{i2}, $i = 1, \dots, k$, satisfy the conditions we need.

Proof of Claim in (a): We claim that the conclusion holds for any normal affine variety X (and factorial $\Rightarrow$ normal). Indeed, we can embed X as an open subvariety into a normal projective variety $\tilde{X}$ (first embed X into $\mathbb{A}^n$, take the closure $\overline{X}$ in $\mathbb{P}^n$, and for $\tilde{X}$ take the normalization of $\overline{X}$). Up to a scalar multiple a rational function on $\tilde{X}$ is determined by its divisor. The divisor of an invertible function in $\mathbb{C}[X]$ is a linear combination of codim 1 components of $\tilde{X} \setminus X$. So $\mathbb{C}[X]^\times / \mathbb{C}^\times$ embeds into a finite rank lattice, and we are done.