HW 4 Solutions

P1. For $A \in \text{Mat}_n(C)$, we have $A^T A \in D_n^+$. Also for $g \in O_n$, have $(gA)^T gA = A^T g^T g A = [g]_2 \circ [id] = A^T A$. So the map $\psi: A \mapsto A^T A: \text{Mat}_n(C) \to D_n$ is $O_n$, inv. Since $D_n^+$ is normal, what we need to check, thx to Igusa's criterion, is that $\psi$ is surjective & every fiber contains a unique closed orbit.

- $\psi$ is surjective: Note that for $B \in \mathfrak{gl}_n$, we have $\psi(AB) = B^T \psi(A) B$. Every $C \in D_n^+$ is $C_n$-conjugate to a matrix of the form $\text{diag}(1, 0, 0, \ldots, 0) = \psi \left( \begin{pmatrix} 1 & \cdots & 0 \\ 0 & \ddots & \vdots \\ \vdots & \ddots & 0 \\ 0 & \cdots & 1 \end{pmatrix} \right)$. This shows that $\psi$ is surjective.

- As in the proof of the fund. then for $C_n$, the proof of the claim that every fiber of $\psi$ contains a single closed orbit reduces to the following two claims: (i) if $\text{GL}(C \cong O_n)$ is closed, then $\text{im} A \oplus \ker A^T = C^n$ (which is equivalent to (ii) $m(A)$ is non-degenerate).

(ii) $C \in D_n^+$, $\psi^*(C)$ contains a unique orbit $C \cong \text{GL}(C \cong O_n)$ non-degenerate.

Proof of (i): let $U_0 = (\text{im} A) \cap (\text{im} A)^\perp \neq 0$. Pick a complement $U_0$ to $U_0$ in $U \& U_0$ to $U_0$ in $U^\perp$. Define a one-parameter subgroup $\gamma: \mathbb{R} \to O_n$ by $\gamma(t)_{|U_0} = t \cdot \text{id}, \gamma(t)_{|U_0} = t \cdot \text{id}$.

We assume that the form $\omega = \omega$. The image of $\gamma$ is indeed in $O_n$ and $\lim \gamma(t) A$ is the compact part of $A + H$. This finishes (i).

Proof of (ii): let $Ah, Ah'$ satisfy $A^T Ah = C \in D_n^+$ & $\text{im} A, \text{im} A'$ are non-degenerate w.r.t. (ii). Since $\text{im} A \oplus \ker A^T = C^n$, we have $\text{rk} A = \text{rk} C$, similarly $\text{rk} A' = \text{rk} C$. It follows that $\text{im} A, \text{im} A'$ are conjugate in $O_n$. We can assume that $\text{im} A = \text{im} A' = U$. Now lets view $A$ as a collection of column vectors $u_1, \ldots, u_n \in U$, similarly get vectors $u_1', \ldots, u_n' \in U$, it is enough to show $\exists g \in O(U)$ in $g v_i = v_i$. Note that $C$ is the Gram matrix for $v_1, \ldots, v_n$ (and for $v_1', \ldots, v_n'$).

We have a diagonal element subset $I \subseteq \{1, \ldots, n\}$ s.t. the corresponding minors in $C$
is nonzero $\Leftrightarrow v_j, i \in I, \& \bar{v}_j \in I, \&$ are bases of $U$. Their Gram matrices are the same so $g \in G(U)$ defined by $g_{ij} = v_i \cdot v_j$ is, actually, in $O(U)$. Since $v_j, j \in I$ is uniquely recovered from $v_i, i \in I,$ and $C$ (and same for $v_j$'s) we see that $g_{ij} = v_i \cdot v_j$. This finishes the proof.

P2. It's enough to show that $C[\text{Mat}_n]^O$ is gen'd by traces of monomials in $A, A^T$. From here and $C[\text{Mat}_n] \otimes \rightarrow C[\text{Mat}_n^+]^O$, it follows that $C[\text{Mat}_n^+]^O$ is gen'd by $\text{Tr}(A^i)$, $i \geq 0$ (on $\text{Mat}_n^+$, have $A = A^T$). By the proof of a Thm in the $G_2$-case, $\text{Tr}(A^i)$ for $i \geq n$ lies in the subalgebra gen'd by $\text{Tr}(A^j), j = 1, \ldots, n$. These cents are alg indep, to see this we can restrict them to diagonal matrices, as in the lecture.

To prove that $C[\text{Mat}_n]^O$ is gen'd by traces of monomials in $A, A^T$, it's enough to check that $(\text{Mat}_n^{\otimes k})^O$ is spanned by products of traces in $B_{c,i}, i = 1, \ldots, n$. Thanks to the presence of a non-degenerate form on $U$, we have $\text{Mat}_n(U) = U \otimes U^* \approx U \otimes U$. The permutation of factors maps $U \otimes U^* \rightarrow U \otimes U$ corresponds, on the level of matrices, to $A \mapsto A^T$. In our case the permutation map $U \otimes U \rightarrow U \otimes U$ is, therefore, $A \mapsto A^T$.

Now identify $\text{Mat}_n^{\otimes k} \cong U^{\otimes k}$. We already know that $(U^{\otimes k})^O$ is gen'd by elements $F_I$, where $F_I(c_{i_1} \otimes \cdots \otimes c_{i_k}) = \prod (c_{i_1}, c_{i_2})$, and $I$ is a partition of $[1, n]$, and the product is taken over all pairs $(i < j)$ in $I$. We claim that $F_I$ gives a required product of traces.

For this it's enough to understand an effect of applying a single pairing: $c_{i_1} \otimes c_{i_2} \mapsto (\hat{u}_{i_1} \otimes \hat{u}_{i_2})$. Since we can permute the matrices, it's enough to understand the situation where $d_1, d_2 \in \{1, 2, 3, 4\}$. We can also assume $d_1 \leq d_2$.

Case 1: $d_1 = 1, d_2 = 2$. We get $B_{d_1} \otimes B_{d_2} \rightarrow \text{Tr}(B_{d_2} B_{d_1} \otimes B_{d_2})$.

Case 2: $d_1 = 2, d_2 = 3$. That's the map $U \otimes U^* \rightarrow U \otimes U^*$ contracting 2nd & 3rd factor. This is the matrix multiplication: $B_{d_1} \otimes B_{d_2} \rightarrow B_{d_2} B_{d_1}$.
Case 2': $d_1 = 1, d_2 = 4$: $B_1 \otimes B_2 \rightarrow B_2 B_1$

Case 3: $d_1 = 1, d_2 = 3$: Thx to $u_1 \otimes u_2 \rightarrow u_2 \otimes u_1$, corrup to $B_2 \mapsto B_1^T$, we reduce to Case 2 and the map is $B_1 \otimes B_2 \rightarrow B_1^T B_2$.

Case 3': $d_1 = 2, d_2 = 4$: Similarly to the previous case, we get $B_1 \otimes B_2 \rightarrow B_1 B_2^T$.

To summarize: applying a single pairing amounts to either taking trace or multiplying matrices (or their transposes). This finishes the proof of an interpretation of $F_2$ on the level of matrices.

P3: For $U$ we take the group of lower triangular matrices.

Note that $f: \mathbb{C}^n \rightarrow \mathbb{C}$ is indeed $U$-invariant. Let $X = C C r(s_k)$.

We claim that $X_f = U \cdot C (e_i, \ldots, e_k)$. Indeed, $X_f$ consists of all $e_i, \ldots, e_k$.

Span$(e_i, \ldots, e_k) \bigcap$ Span$(e_i, \ldots, e_k) = 0$.

Applying a unimodular transformation to $e_i, \ldots, e_k$, we can assume that $e_i = e_i + \sum a_j e_j, \quad e_k = e_k + \sum a_j e_j, \quad i < k$.

Clearly, there exists $g \in U$ such that $e_i, \ldots, e_k = g\cdot e_i, \ldots, e_k$.

Thus, $X_f = C [x]^u \rightarrow \mathbb{C}[x]^u$, which proves our statement. So $C [x]^u \rightarrow \mathbb{C}[x]^u \rightarrow \mathbb{C}[x]^u \rightarrow \mathbb{C}[x]^u \rightarrow \mathbb{C}[x]^u$.

Also, $C[f] \subset C[x]^u \subset C[f^u]$. Assume $C[x]^u$, which is nonsense. $f$ is clearly non-invertible ($f(0) = 0$).
P4: a) Note that \((\mathbb{C}[[u]] \otimes B)^T\) is fin. gen. graded \(G\)-algebra & 
\[(\mathbb{C}[[u]] \otimes B)^T)^u = (\mathbb{C}[[u]]^u \otimes B)^T = (\mathbb{C}[x^+] \otimes B)^T = B.\] So it remains to check that every graded \(G\)-algebra has the form \((\mathbb{C}[[u]] \otimes B)^T\) for a graded algebra \(B\). Note that \(V(\lambda + \mu)\) occurs in \(V(\lambda) \otimes V(\mu)\) w. mult. = 1 and this copy of \(V(\lambda + \mu)\) is gensd by \(v_\lambda \otimes v_\mu\). It follows that the product \(A^u \otimes A^u \rightarrow A^{u+u}\) is completely recovered from its restriction to \(A^u \otimes A^u \rightarrow A^{u+u}\).

Now note that such restrictions for \(A\) and for \((\mathbb{C}[[u]] \otimes A^u)^T\) coincide. This establishes a \(G\)-equiv. iso \(A \rightarrow (\mathbb{C}[[u]] \otimes A^u)^T\).

b) The claim that \(A_{\lambda; i}\) is an algebra filtration follows from the observation that all highest weights that appear in \(V(\lambda) \otimes V(\mu)\) are \(\lambda + \mu\). Note that \(A^u = (gr A)^u\) so \((gr A)^u\) is fin. gensd. \(\Rightarrow gr A\) is fin. gen.\