Lec 9 write-ups

Thm 2 (Dynkin) Let (e,h,f), (e',h,f') be two $\mathfrak{sl}_2$-triples. Then $\exists$ $g\in Z_G(h)$ s.t. $ge=e'$ ($\Rightarrow gf=f'$)

Proof: $G_0 := Z_G(h) \curvearrowright \mathfrak{g}_2 = \{x\in\mathfrak{g} \mid [h,x]=2x\}$; $T_e G_0 e = [\mathfrak{g}_0, e] =$ [(ii) of $\mathfrak{sl}_2$-lemma] $= \mathfrak{g}_2 \Rightarrow G_0 e \subset \mathfrak{g}_2$ is open. But $G_0 e' \subset \mathfrak{g}_2$ is open for similar reason, hence $G_0 e \cap G_0 e' \neq \phi \Rightarrow G_0 e = G_0 e'$ $\square$

Thm 6: # nilp orbits $<\infty$

Proof: For $\mathfrak{h} = \mathfrak{gl}_n$, the proof follows from JNF (nilp orbits $\xleftrightarrow{\sim}$ partitions of n). For gen'l $\mathfrak{g}$: $\mathfrak{g} \hookrightarrow \mathfrak{h} = \mathfrak{gl}_n$ & for $x\in\mathfrak{g}$, x is nilp in $\mathfrak{g} \Leftrightarrow$ it's nilp in $\mathfrak{gl}_n$. By Lem 1 (compare to the proof of Prop 1), Gx is conn'd comp't of $Hx \cap \mathfrak{g}$. There are fin many possible nilp't Hx & fin. many conn'd comp's. So # of nilp't Gx is finite $\square$