Inv. thy, HW 2 solutions

1) Let's show there are no non-const. alg. variety morphisms $G_1 \to G_2$. Since G_1 is conn'd we can assume G_2 is conn'd. A morphism $\varphi: G_1 \to G_2$ gives rise to $\varphi^*: \mathbb{C}[G_2] \to \mathbb{C}[G_1]$. But $\mathbb{C}[G_2] = \mathbb{C}[y_1^{\pm 1}, \dots, y_k^{\pm 1}]$ and $\mathbb{C}[G_1] = \mathbb{C}[x_1, \dots x_\ell]$. The only invertible elements in the latter are constant. So $\varphi^*(y_i) \in \mathbb{C}$ and φ is constant.

Let's show there are no nontrivial alg. group homom'ms $G_2 \xrightarrow{\varphi} G_1$. Note that G_1 contains no finite order elements so all finite order elem's in G_2 go to 1. The connected component G_2° is a torus, the finite order el'ts are dense there. So $\varphi(G_2^\circ) = \{1\}$ and φ factors through $G_2/G_2^\circ \to G_1$. Again, since there are no finite order el'ts in G_1, this homomorphism is trivial.

2) Consider the action of H on Z by conjugation, it permutes the el'ts of Z. Since H is connected, any permutation appearing in this way is trivial.

3) First, let us reduce to the case when x is nilpotent (in the Lie alg'e of a reductive alg'c group). Decompose x as $x_s + x_n$. We know that $Z_G(x_s)$ is reductive & $Z_G(x) = Z_G(x_s) \cap Z_G(x_n)$. In other words, $Z_G(x)$ coincides w. the centralizer of x_n in $Z_G(x_s)$. We replace (G, x) w. $(Z_G(x_s), x_n)$ and assume x is nilpotent.

Now let H be the centralizer of x in G, reductive by our assumption. We know that $Z_G(H)$ is reductive. So we can replace G w. $Z_G(H)^\circ$ and assume that the centralizer of x in G coincides w. the center (and similarly in $\mathfrak{g}$). The center consists of s/simple elements. But x lies in $\mathfrak{z}_{\mathfrak{g}}(x)$ and is nilpotent. So $x=0$, which implies the claim.

$\square$

Solution to extra-credit problem (HW 2)

Recall that the nilpotent orbits are in bijection with the conj'y classes of $\mathfrak{sl}_2$-triples. So (a) means that if two orthogonal/symplectic rep's of $\mathfrak{sl}_2$ are isomorphic, then there is an orthogonal/symplectic isomorphism. Let V be such a representation, and let U_n be the multiplicity space of the $n+1$-dimensional irrep V_n in V so that $V = \bigoplus_n V_n \otimes U_n$. It's easy to see that two different $V_n \otimes U_n$, $V_m \otimes U_m$ are orthogonal and, of course, any isomorphism $V \to V$ preserves the decomp'n $V = \bigoplus_n V_n \otimes U_n$. Now note that the tensor product of two spaces w. orthogonal/symplectic forms carries a natural form that is orthogonal if both forms are orth'l or both forms are sympl'c and is sympl'c if one of the forms is orthogonal and the other is symplectic. Note that $V_n \cong S^n(\mathbb{C}^2)$ is orthogonal if n is even and is symplectic when n is odd. So $U_n = \mathrm{Hom}_G(V_n, V) = (V_n \otimes V)^G$ carries a natural non-degenerate form. If

(a) V, V_n are both orthogonal or both symplectic, then U_n is orthogonal

(b) If one of V_n, V is orthogonal and the other is sympl'c, then U_n is symplectic, hence even dimensional.

This implies (b). To prove (a), note that $\mathrm{Hom}_G(V_n \otimes U_n, V_n \otimes U_n) = \mathrm{Hom}(U_n, U_n)$. There is an isomorphism $U_n \to U_n$ preserving the form, hence there's such an isomorphism in $\mathrm{Hom}_G(V_n \otimes U_n, V_n \otimes U_n)$.

Let's prove (c). The O_n-orbit of an $\mathfrak{sl}_2$-triple (e,h,f) splits into two SO_n-orbits $\Leftrightarrow$ the centralizer $Z_{O_n}(e,h,f) \subset SO_n$. This centralizer is of the form $\prod_n \{\pm \mathrm{Id}_{V_n}\} \times G(U_n)$, where $G(U_n)$ is the group of the form preserving automorphisms of U_n. If $V_n \neq 0$ for $n+1$ odd, then $\exists\, g \in G(U_n) = O(U_n)$ s.t. $\det(-\mathrm{Id}_{V_n} \otimes g) = -1$ & $Z_{O_n}(e,h,f) \not\subset SO_n$. But if $V_n \neq 0 \Rightarrow n+1$ even $\Rightarrow G(U_n) = Sp(U_n) \Rightarrow Z_{O_n}(e,h,f) \subset SO_n$. This proves (c)