CHEREDNIK ALGEBRAS AT $T = 0$ AND CALOGERO-MOSER SPACES

IVAN LOSEV, WRITTEN BY SETH SHELLEY-ABRAHAMSON

Abstract. These are notes for a talk in the MIT-Northeastern Spring 2015 Geometric Representation Theory Seminar. We discuss rational Cherednik algebras at $t = 0$.

Contents

1. Calogero-Moser Spaces

We use notation from the previous lecture, with $\mathfrak{h} := V$. In particular, $\mathfrak{h}$ is a finite-dimensional $\mathbb{C}$-vector space, and $W \subset GL(\mathfrak{h})$ is a finite subgroup generated by complex reflections. We have the $\mathbb{C}[\mathcal{C}]$-algebra $\mathbb{H}_0 = \mathbb{H}/(T)$, the generic rational Cherednik algebra at $T = 0$. It is naturally $\mathbb{Z}$-graded with $\deg \mathfrak{h}^* = \deg W = 0$, $\deg \mathfrak{h} = \deg C_s = 1$. One has other gradings as well, for example, $\deg \mathfrak{h} = \deg W = 0$ and $\deg \mathfrak{h}^* = 1$. For a $c \in \mathcal{C}$, we have the associated specialization $\mathbb{H}_{0,c}$, a filtered $\mathbb{C}$-algebra.

We briefly recall a few results from Yi's talk.

a) We have the PBW theorem, stating that the natural $\mathbb{C}[\mathcal{C}]$-linear multiplication map

$$\mathbb{C}[\mathcal{C}] \otimes S\mathfrak{h}^* \otimes \mathbb{C}W \otimes S\mathfrak{h} \to \mathbb{H}_0$$

is an isomorphism.

b) Let $\mathcal{Z} = Z(\mathbb{H}_0)$ denote the center of $\mathbb{H}_0$, and similarly let $\mathcal{Z}_c = Z(\mathbb{H}_{0,c})$ denote the center of $\mathbb{H}_{0,c}$. We have the Satake isomorphism, which states that the maps

$$\mathcal{Z} \to e\mathbb{H}_0 e$$

and

$$\mathcal{Z}_c \to e\mathbb{H}_{0,c} e$$

given by $z \mapsto ze$ are isomorphisms. The algebra $\mathcal{Z}_c$ is filtered, as the specialization of the graded algebra $\mathcal{Z}$, and we have

$$\mathrm{gr}\mathcal{Z}_c = \mathcal{Z}_0 = S(\mathfrak{h} \oplus \mathfrak{h}^*)^W.$$

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c) Let $\delta = (\prod_{s\in S}\alpha_s)^? \in S\mathfrak{h}^{*W}$ denote a W-invariant power of the discriminant element, so that $\mathfrak{h}^{reg}$ is the associated principal open set. Then we have the localization lemma, stating that the following diagram of $\mathbb{C}[\mathcal{C}]$-algebras commutes, where the diagonal map is given by the Dunkl operator embedding:

$$\begin{tikzcd} \mathbb{H}_0 \arrow[r] \arrow[dr] & \mathbb{H}_0[\delta^{-1}] \arrow[d, "\cong"'] \\ & \mathbb{C}[\mathcal{C}]\otimes\mathbb{C}[T^*\mathfrak{h}^{reg}]\rtimes\mathbb{C}W \end{tikzcd}$$

All maps here are injective maps of algebras, with the vertical map an isomorphism

c*) Similarly, setting $\delta^* = (\prod_{s\in S}\alpha_s^\vee)^? \in S\mathfrak{h}^{W}$, we have the diagram

$$\begin{tikzcd} \mathbb{H}_0 \arrow[r] \arrow[dr] & \mathbb{H}_0[\delta^{*-1}] \arrow[d, "\cong"'] \\ & \mathbb{C}[\mathcal{C}]\otimes\mathbb{C}[T^*\mathfrak{h}^{*reg}]\rtimes\mathbb{C}W \end{tikzcd}$$

Let $X = \mathrm{Spec}(\mathcal{Z})$. Then there is a natural flat map $X \to \mathcal{C}$, and $X_c := \mathrm{Spec}(\mathcal{Z}_c)$ is the fiber above $c \in \mathcal{C}$. We call X and X_c *Calogero-Moser spaces.*

Note by PBW we have an embedding $S\mathfrak{h}^W \otimes S\mathfrak{h}^{*W} \hookrightarrow \mathbb{H}_0$. Even better:

Lemma 1. *$S\mathfrak{h}^W \otimes S\mathfrak{h}^{*W} \subset \mathcal{Z}$.*

Proof. That $S\mathfrak{h}^{*W} \subset \mathcal{Z}$ follows immediately from the Dunkl operator embedding. The case of $S\mathfrak{h}^W$ is similar, using the diagram **c***) above. □

So, we see $\mathcal{Z} \supset \mathbb{C}[\mathcal{C}] \otimes S\mathfrak{h}^{*W} \otimes S\mathfrak{h}^W$.

Lemma 2. *$\mathcal{Z}$ is free graded rank $|W|$ module over $\mathbb{C}[\mathcal{C}] \otimes S\mathfrak{h}^{*W} \otimes S\mathfrak{h}^W$.*

Proof. We see that $S(\mathfrak{h}\oplus\mathfrak{h}^*)^W$ is a direct summand of $S\mathfrak{h}\otimes S\mathfrak{h}^*$ as a $S\mathfrak{h}^W\otimes S\mathfrak{h}^{*W}$-module (in fact, as a $S(\mathfrak{h}\oplus\mathfrak{h}^*)^W$-modules) - indeed, a complement is given by $(1-e)(S\mathfrak{h}\otimes S\mathfrak{h}^*)$ where $e = \frac{1}{|W|}\sum_{w\in W} w \in \mathbb{C}W$. By the Chevalley theorem, $S\mathfrak{h}\otimes S\mathfrak{h}^*$ is a free module over $S\mathfrak{h}^W \otimes S\mathfrak{h}^{*W}$, and hence $S(\mathfrak{h}\otimes\mathfrak{h}^*)^W$ is a finitely generated projective, hence free, module over $S\mathfrak{h}^W\otimes S\mathfrak{h}^{*W}$. The rank is $|W|$, as one can see for example by considering the fiber at a point of $(\mathfrak{h}\times\mathfrak{h}^*)^{reg}$. This gives the result for the specialization at $c = 0$. The result then follows from a version of Nakayama's lemma and the facts that $\mathcal{Z}$ is $\mathbb{Z}_{\geq 0}$-graded with $\deg\mathcal{C} = 1$ and that $\mathcal{Z}$ is free over $\mathbb{C}[\mathcal{C}]$. □

So we see that the natural maps $X \to \mathcal{C}\times\mathfrak{h}/W\times\mathfrak{h}^*/W$ and the specialization $X_c \to \mathfrak{h}/W\times\mathfrak{h}^*/W$ are finite degree-$|W|$ maps. In view of **c**) and **c***) we see over $\mathcal{C}\times\mathfrak{h}^{reg}/W\times\mathfrak{h}^*/W$, X is $\mathcal{C}\times(\mathfrak{h}^{reg}\times\mathfrak{h}^*)/W$, and similarly over $\mathcal{C}\times\mathfrak{h}/W\times\mathfrak{h}^{*reg}/W$, X is $\mathcal{C}\times(\mathfrak{h}\times\mathfrak{h}^{*reg})/W$.

2. Poisson Structure

We define a $\mathbb{C}[\mathcal{C}]$-linear Poisson bracket on $\mathcal{Z}$ as follows. Let $\iota : \mathcal{Z} \to \mathbb{H}$ be any $\mathbb{C}[\mathcal{C}]$-linear lift of the inclusion $\mathcal{Z} \to \mathbb{H}_0$ to the algebra $\mathbb{H}$:

$$\begin{tikzcd} & \mathbb{H} \arrow[d, "/(T)"] \\ \mathcal{Z} \arrow[ur, "\iota"] \arrow[r] & \mathbb{H}_0 \end{tikzcd}$$

By the PBW theorem, a lift $\mathbb{H}_0 \to \mathbb{H}$ exists, giving the existence of a lift ι. Given $a, b \in \mathcal{Z}$ and $h \in \mathfrak{h}$, that a is central in $\mathbb{H}_0$ implies $[\iota(a), h], [\iota(b), h] \in T\mathbb{H}$. It follows from the Jacobi identity that $[[\iota(a), \iota(b)], h] \in T^2\mathbb{H}$. It follows that $[\iota(a), \iota(b)] \in T\iota(\mathcal{Z}) + T^2\mathbb{H}$. We then define

$$\{a, b\} := \frac{1}{T}[\iota(a), \iota(b)] \bmod (T)$$

It is an exercise to see that this is independent of the lift ι, that $\{\cdot,\cdot\}$ is Poisson, and of degree -1. It is easy to see that $\{\cdot,\cdot\}$ vanishes on $S\mathfrak{h}^W$ and $S\mathfrak{h}^{*W}$.

Definition 3. *Let X be an affine Poisson variety. A closed subscheme $Y_0 \subset X$ is called Poisson, if $\{\mathbb{C}[Y], I(Y_0)\} \subset I(Y_0)$, where we write $I(Y_0)$ for the ideal of Y_0.*

Definition 4. *We say a Poisson variety Y has finitely many leaves if Y has a finite stratification $Y = \coprod_i Y_i$ into locally closed subvarieties such that Y_i is symplectic and $\overline{Y_i}$ is Poisson. We call the Y_i the symplectic leaves of Y.*

Exercise 5. *Let V be a symplectic vector space, $\Gamma \subset Sp(V)$ a finite subgroup, and $Y = V/\Gamma$. Then the symplectic leaves of Y are in bijection with the conjugacy classes of stabilizers Γ_v of Γ, with the conjugacy class $[\Gamma']$ corresponding to the image in V/Γ of $\{v \in V : \Gamma_v = \Gamma'\}$.*

Exercise 6. *$Y^{sing} \subset Y$ is Poisson. Also, the nonsymplectic locus in a smooth Poisson variety is a Poisson subvariety.*

Proposition 7. *X_c has finitely many symplectic leaves.*

Proof. The Poisson structure on $Z_0 = S(\mathfrak{h} \oplus \mathfrak{h}^*)^W$ is the standard Poisson structure restricted from $S(\mathfrak{h} \oplus \mathfrak{h}^*)$. Indeed, the bracket is independent of the lift $Z_0 \to \mathbb{H}_{T,0}$. The standard Poisson bracket on $S(\mathfrak{h} \oplus \mathfrak{h}^*)^W$ comes from a lift

$$\iota_0 : \mathcal{Z}_0 \to \mathcal{D}_T(\mathfrak{h})^W \subset \mathbb{H}_{T,0} = \mathcal{D}_T(\mathfrak{h}) \rtimes \mathbb{C}W.$$

It follows that the zero fiber of the scheme $X_{\mathbb{C}c}/\mathbb{C}c$ is generically symplectic, so X_c is generically symplectic. Now let $Y \subset X_c$ be an irreducible Poisson subvariety. It is enough to show Y is generically symplectic. Indeed, by the previous exercise, the locus, where Y is not smooth or not symplectic is a proper Poisson subvariety and we can apply the induction on dimension of Y.

Let $I \subset \mathbb{C}[X_c]$ be the ideal of Y. The algebra $\mathbb{C}[X_{\mathbb{C}c}]$ is the Rees algebra $R_t(\mathbb{C}[X_c]) = \bigoplus_{i \geq 0} \mathbb{C}[X_c]_{\leq i} t^i$. Consider the subscheme $Y_{\mathbb{C}c} \subset X_{\mathbb{C}c}$ defined by $R_t(I)$. Its zero fiber is a Poisson subscheme in $X_0 = (\mathfrak{h} \oplus \mathfrak{h}^*)/W$ and hence has finitely many leaves. In particular, it is generically symplectic. So Y_c is generically symplectic. $\square$

3. Smoothness

It is natural to ask whether X_c can be smooth, and to characterize the smooth points.

Let us address the smoothness. In the case $W = G(\ell, 1, n)$ (this is the group $S_n \ltimes (\mathbb{Z}/\ell\mathbb{Z})^n$ acting on $\mathbb{C}^n$) all X_c are Nakajima quiver varieties, and X_c is smooth for generic c. There is exactly one more exceptional complex reflection group (known as G_4) where a generic Calogero-Moser space is smooth.

$\mathbb{H}_{0,c}e$ is a finitely generated $\mathcal{Z}_c$-module, so we can view it as a coherent sheaf on X_c. Now consider the subvariety

$$X_c^{sph} := \{x \in X : \dim(\mathbb{H}_{0,c}e)_x \text{ is minimal}\}.$$

This is a nonempty Zariski open subset. Over $\mathfrak{h}^{reg}/W \times \mathfrak{h}^*/W$, X_c is $(\mathfrak{h}^{reg} \times \mathfrak{h}^*)/W$. There, $\mathbb{H}_{0,c}e$ is $\mathbb{C}[T^*\mathfrak{h}^{reg}]$, which is a free rank-$|W|$ module over $\mathbb{C}[T^*\mathfrak{h}^{reg}]^W$. So we see the minimal fiber dimension of $\mathbb{H}_{0,c}e$ is $|W|$.

Theorem 8. $X_c^{sph} = X_c^{smooth}$.

For $x \in X_c^{sph}$, we have the associated maximal ideal $\mathfrak{m}_x \subset \mathcal{Z}_c$, and one has $\mathbb{H}_{0,c}/\mathbb{H}_{0,c}\mathfrak{m}_x = \mathrm{Mat}_{|W|}(\mathbb{C})$ by considering the isomorphism $\mathbb{H}_{0,c} \cong \mathrm{End}_{\mathcal{Z}_c}(\mathbb{H}_{0,c}e)$ from Yi's talk.

Proof. First we show $X_c^{smooth} \subset X_c^{sph}$. We need to show that all fibers of $\mathbb{H}_{0,c}e$ have dimension $|W|$ on X_c^{smooth}, and for this it suffices to produce a (local) connection for $\mathbb{H}_{0,c}e$ over this open set. To this end, given a section s of $\mathbb{H}_{0,e}$ and ξ some vector field, we define $\nabla_\xi s$ as follows. First note that, since we are only interested in constructing the connection locally, i.e, in a neighborhood of each point, it suffices to assume ξ is a Hamiltonian vector field $v(f)$ for some given function f. Indeed, X^{smooth} is symplectic by Proposition 7 so every point has a neighborhood where one can choose a basis of vector fields consisting of Hamiltonian vector fields and it is enough to define the connection on the basis elements. Let $\iota_c : \mathcal{Z}_c \to \mathbb{H}_{T,c}$ be a lift as in the previous section, and pick a lift $\tilde{s}$ of the section s in $\mathbb{H}_{T,c}e$. Then define

$$\nabla_\xi s := \frac{1}{T}[\iota_c(f), \tilde{s}] \bmod (T).$$

Note that the right hand side is independent of $\tilde{s}$ (but, generally, depends on ι).

Now we show the reverse inclusion $X_c^{sph} \subset X_c^{smooth}$. Recall that an affine variety Y is smooth if and only if $\mathbb{C}[Y]$ has finite homological dimension. Observe also that we have the following bound on homological dimension

$$\mathrm{HomDim}\ \mathbb{H}_{t,c} \leq 2\dim_{\mathbb{C}} \mathfrak{h} < \infty$$

This follows from the fact that $\mathrm{gr}\mathbb{H}_{t,c} = S(\mathfrak{h} \oplus \mathfrak{h}^*) \rtimes \mathbb{C}W$ has homological dimension $2\dim \mathfrak{h}$, and that the homological dimension is does not increase under filtered deformations. So, for $f \in \mathcal{Z}_c$, $\mathbb{H}_{0,c}[f^{-1}]$ always has finite homological dimension. If $X_{c,f}$, the principal open subset associated to f, is contained in X_c^{sph} then $\mathbb{H}_{0,c}[f^{-1}]$ is $\mathrm{End}_{\mathcal{Z}_c[f^{-1}]}(\mathbb{H}_{0,c}e[f^{-1}])$, a vector bundle. But then we see $\mathbb{H}_{0,c}[f^{-1}]$ is Morita equivalent to $\mathcal{Z}_{0,c}[f^{-1}]$, so the latter has finite homological dimension and $X_{c,f}$ is smooth, as needed. $\square$