Calogero-Moser spaces vs quiver varieties

1) Affine Nakajima quiver varieties

2) Rel-n to CM spaces

1) Q-quiver (=oriented graph). Formally, a quadruple (Q_0, Q_1, t, h), $t, h: Q_1 \to Q_0$, Q_0 - vertices, Q_1-arrows: $1 \circ \xrightarrow{a} \circ 2$, $t(a)=1$, $h(a)=2$

Can define framed representation space $R := R(Q, v, w)$; $v, w \in \mathbb{Z}_{\geq 0}^{Q_0}$ -dimension and framing vectors, $V_k = \mathbb{C}^{v_k}$, $W_k := \mathbb{C}^{w_k}$, $k \in Q_0$

$R := \bigoplus_{a \in Q_1} Hom(V_{t(a)}, V_{h(a)}) \oplus \bigoplus_{k \in Q_0} Hom(W_k, V_k)$

comes with an action of $GL(v) = \prod_{k \in Q_0} GL(V_k)$

(change of bases in the spaces V_k)

Consider $T^*R = R \oplus R^* = [\text{tr. pairings}] = \bigoplus_{a \in Q_1} (Hom(V_{t(a)}, V_{h(a)}) \oplus$

(*) $Hom(V_{h(a)}, V_{t(a)})) \oplus \bigoplus_{k \in Q_0} (Hom(W_k, V_k) \oplus Hom(V_k, W_k))$

A typical element in T^*R will be written as (A_a, B_a, i_k, j_k)

T^*R comes with a canonical symplectic form, say ω, that is G-stable. So the G-action on T^*R admits a moment map a G-equiv. morphism $\mu: T^*R \to \mathfrak{g}^*$ such that $\{\mu^*(\xi), \cdot\} = \xi_{T^*R}$ $\forall \xi \in \mathfrak{g}$. This map is given by $\xi \mapsto \xi_R \in Vect(R) \hookrightarrow \mathbb{C}[T^*R]$

We will need a linear algebraic interpretation of μ that is left as an exercise (use def-n of μ^* & identification by trace pairings)

Lem: Using the interpretation of T^*R in (*) we get

$\mu(A_a, B_a, i_k, j_k) = \sum_{a \in Q_1} (A_a B_a - B_a A_a) + \sum_{k \in Q_0} i_k j_k \; (\in \mathfrak{g}^* = \bigoplus_{k \in Q_0} \mathfrak{gl}(V_k))$

We define the scheme $\mathcal{M}_\lambda(v, w) = \mu^{-1}(\lambda) /\!/ G$. Here $\lambda \in (\mathfrak{g}^*)^G \simeq \mathbb{C}^{Q_0}$ and $/\!/$ means the categorical quotient under the G-action. If a reductive group G acts on an affine scheme X, then $X /\!/ G := Spec(\mathbb{C}[X]^G)$

a.k.a δ

2) Rel-n to CM spaces:

2.1) $W=\mathbb{Z}/\ell\mathbb{Z}$, Q-cyclic quiver w. ℓ vertices

$v=(1,\dots,1)$
$w=(0,0,\dots,0)$

$T^*R=\mathbb{C}^{2\ell}=\{(x_1\dots x_\ell, y_1\dots y_\ell): \underset{i}{\bullet}\underset{x_i}{\overset{y_i}{\rightleftarrows}}\underset{i+1}{\bullet}\}$, $G=(\mathbb{C}^\times)^\ell$, $(t_i).(x_i,y_i):=$
$=(t_{i+1}x_it_i^{-1}, t_iy_it_{i+1}^{-1})$

$\mu=(x_1y_1-x_\ell y_\ell, x_2y_2-x_1y_1, \dots, x_{\ell-1}y_{\ell-1}-x_\ell y_\ell)$

$(\mathbb{C}^\times)^\ell$-invariant functions on $\mu^{-1}(\lambda)$: $F=x_1\dots x_\ell$, $G=y_1\dots y_\ell$, $H=x_1y_1$ $(=x_2y_2=\dots=x_\ell y_\ell)$
generate $\mathbb{C}[\mu^{-1}(\lambda)]^G$. Rel-n: for $\lambda=0$: $FG=H^\ell \rightsquigarrow \mathbb{C}[\mathbb{C}^2/(\mathbb{Z}/\ell\mathbb{Z})]$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \mathbin{\rotatebox{90}{=}}\ \mathbb{C}[\mu^{-1}(0)]^G$

Reminder: RCA $\eta=1+\ell\mathbb{Z}$ - generator $\mathbb{Z}/\ell\mathbb{Z}$

$$H_{t,c}=\mathbb{C}\langle x,y\rangle / ([y,x]=t+\sum_{i=1}^{\ell-1}c_i\eta^i)$$

Let $\pi_i\in\mathbb{C}W$ - idemp corresp to irred $\eta\mapsto\eta^i$ write $t+\sum_{i=1}^{\ell-1}c_i\eta^i$
as $\sum_{i=0}^{\ell-1}\lambda_i\pi_i$, e.g. $t=\sum_{i=0}^{\ell-1}\lambda_i$

The condition that $H_{t,c}$ admits a module $\cong_W \mathbb{C}W$:
$\mathrm{tr}\,[y,x]=0 \Rightarrow \sum_{i=0}^{\ell-1}\lambda_i=0$ $(\Leftrightarrow \mu^{-1}(\lambda)\neq\emptyset)$

Now consider the representation variety $\mathrm{Rep}(H_{0,c},\mathbb{C}W)$ parameterizing semisimple reps of $H_{0,c}$ in $\mathbb{C}W$ that give the representation by left multiplications on $\mathbb{C}W$. This is the quotient of the algebraic variety $\mathrm{Hom}_{\mathbb{C}W\text{alg}}(H_{0,c},\mathrm{End}(\mathbb{C}W))$ by the action of $GL(\mathbb{C}W)^W$.
We have a morphism $\mathrm{Rep}(H_{0,c},\mathbb{C}W)\longrightarrow X_c=\mathrm{Spec}(eH_{0,c}e)$, $V\mapsto eV$.
One can show that this morphism is finite & birational. Since X_c is normal, it is an isomorphism. On the other hand $\mathrm{Rep}(H_{0,c},\mathbb{C}W)$ just is $M_\lambda(\delta,0)$. Indeed, let V_i be the isotypic component of $\eta\mapsto\eta^i$ in $\mathbb{C}W(=\mathbb{C}^\ell)$ so that $\dim V_i=1$. Then $x\in H_{0,c}$ gives rise to operators $x_i: V_i\to V_{i+1}$, while y gives rise to $y_i: V_{i+1}\to V_i$. Then relation $[x,y]=\sum_i\lambda_i\pi_i$ precisely gives the moment map condition

2.2) $W=\mathfrak{S}_n\ltimes(\mathbb{Z}/\ell\mathbb{Z})^n$. The quiver is the same, but the dimensions are different: $v=(n,\dots,n)$, $w=(0,0,\dots,1)$. ~~The correspondence between parame~~

Let us describe the correspondence between parameters. The algebra $H_{c_0,c}$ has two parameters: c_0 corresponding to the conjugacy class of reflections in $\mathfrak{S}_n$ and $\lambda^1_1, \dots, \lambda^1_\ell$ summing to 0 (and corresponding to reflections in $\mathbb{Z}/\ell\mathbb{Z}$). Then the formula for $\lambda_1, \dots, \lambda_\ell$ is

$$\lambda_i = \lambda^1_i/\ell, \quad i=1,\dots,\ell-1, \quad \lambda_\ell = \lambda^1_\ell/\ell - c_0/2$$

(up to normalizations of $c_0, \lambda^1_1, \dots, \lambda^1_\ell$). For the proof in the $\ell=1$ case the reader is referred to Jose's notes from Spring 2014. The case of arbitrary ℓ is similar.