$\mathcal{O}$-seminar, Oct 28: Soergel bimodules vs deformed cat'y $\mathcal{O}$

0) Recap
1) Completed Soergel bimodules
2) Deformed categories $\mathcal{O}$
3) Deformed $\mathbb{V}$-functor

0) Main result of Mitya's talk was an equivalence $\mathcal{O}_0$-proj $\to SMod_{ungr}$ via Soergel's functor $\mathbb{V} := Hom(P_{min}, \bullet)$. Here $SMod_{ungr}$ is the category whose objects are Soergel modules and the morphisms are all $\mathbb{C}[\mathfrak{h}^*]$-linear homomorphisms. In part'r, this classifies indecomposables in SMod: $S_w (w\in W)$

An important step in Mitya's construction was to present $\mathbb{V}$ as the extended translation functor $\tilde{T}_{0\to -\rho}: \tilde{\mathcal{O}}_0 \to \tilde{\mathcal{O}}_{-\rho}$. Namely, consider $\tilde{\mathcal{U}} = \mathcal{U}(\mathfrak{g}) \otimes_{\mathbb{C}[\mathfrak{h}^*]^W} \mathbb{C}[\mathfrak{h}^*]$, where $\mathbb{C}[\mathfrak{h}^*]^W \hookrightarrow \mathcal{U}(\mathfrak{g})$ via the HC isom'm w the center. Let m_λ denote the maximal ideal of λ in $\mathbb{C}[\mathfrak{h}^*]$. Then we have the category $\tilde{\mathcal{O}}_\lambda$ of all $\tilde{\mathcal{U}}$-modules w. locally nilpotent action of $\mathcal{I}_\lambda$ that lie in $\mathcal{O}$ when viewed as $\mathcal{U}(\mathfrak{g})$-modules. Using $\tilde{T}_{0\to -\rho}$ one could see that $End(P_{min}) = C (:= \mathbb{C}[\mathfrak{h}^*]/(\mathbb{C}[\mathfrak{h}^*]^W)$-inv's for the usual action)

Boris has used this classification to classify the indecomposable Soergel bimodules. $\exists!\, B_w \in SBim$ w $B_w \otimes_R \mathbb{C} \xrightarrow{\sim} S_w$, this has yielded an iso'm $K_0(SBim) \xrightarrow{\sim} \mathcal{H}$. A key ingredient in the class'n of indecomposables is the following result of Soergel

Prop 0: $\forall B_1, B_2 \in SBim$: $Hom_{R\otimes R}(B_1, B_2) \otimes_R \mathbb{C} \xrightarrow{\sim} Hom_R(B_1 \otimes_R \mathbb{C}, B_2 \otimes_R \mathbb{C})$

In this talk, we'll explain a proof of this result based on considering deformed categories $\mathcal{O}$ and the deformed Soergel functor

1) Recall that every $B \in SBim$ is a $R \otimes_{R^W} R$-module. Set $\hat{R} = \mathbb{C}[[\mathfrak{h}^*]]$, $\pi: \hat{R} \to \mathbb{C}$ loc. alg. w

the completion of R at 0, a formal power series algebra. We have the completion functor $\bullet^\wedge$: R-mod $\to \hat{R}$-mod (the cat's of fin. gen'd modules) this functor is exact. Note that $\hat{R}\otimes_{R^W} R = \hat{R}\otimes_{\hat{R}^W}\hat{R}$ so for $B\in SBim$ $B^{\wedge_0}$ is an $\hat{R}$-bimodule. Also note that $\bullet^{\wedge_0}: R\otimes_{R^W} R\text{-mod} \to \hat{R}\otimes_{\hat{R}^W}\hat{R}\text{-mod}$ is a tensor functor. Note that since $\hat{R}$ is flat over R, for $B_1, B_2 \in SBim$, we have $Hom_{\hat{R}\text{-}\hat{R}}(B_1^{\wedge_0}, B_2^{\wedge_0}) \xleftarrow{\sim} Hom_{R\text{-}R}(B_1, B_2)^{\wedge_0}$. So Prop 0 is equiv't to Prop 0': $Hom_{\hat{R}\text{-}\hat{R}}(B_1^{\wedge_0}, B_2^{\wedge_0})\otimes_{\hat{R}}\mathbb{C} \xrightarrow{\sim} Hom_R(B_1\otimes_R\mathbb{C}, B_2\otimes_R\mathbb{C})$

21) Categories $\hat{\mathcal{O}}_\lambda$ ($\lambda+\rho$ dominant)

Consider the $\hat{R}$-algebra $\mathcal{U}_{\hat{R}} = \hat{R}\otimes U(\mathfrak{g})$ and the category $\hat{\mathcal{O}}$ of its modules M satisfying the following two conditions:

(a) $\mathfrak{n}_+ \curvearrowright M$ locally nilpotently

(b) $M = \bigoplus_{\lambda\in\Lambda} M_{\hat\lambda}$, where $M_\lambda = \{m\in M \mid \alpha_i^\vee m = (\langle\lambda,\alpha_i^\vee\rangle + z_i)m$, where $z_i \in \hat{R}$ is the element of $\hat{R}$ corresponding to $\alpha_i^\vee\}$

Ex (deformed Verma module) $\hat{\Delta}(\mu) = \mathcal{U}_{\hat{R}} / \mathcal{U}_{\hat{R}}(\alpha_i^\vee - \langle\mu,\alpha_i^\vee\rangle - z_i$, $i=1..rk\,\mathfrak{g}$; $y\in\mathfrak{n}_+)$

Note that the integral part of the category $\mathcal{O}$ is the full subcategory of all objects $M\in\hat{\mathcal{O}}$ where all z_i act by 0, $M\in\hat{\mathcal{O}} \Rightarrow M\otimes_{\hat{R}}\mathbb{C}\in\mathcal{O}_{int}$

The category $\hat{\mathcal{O}}$ inherits many basic properties of $\mathcal{O}_{int}$: weights are bounded by above, weight spaces are finitely generated $\hat{R}$-modules. Moreover, $V\otimes\bullet$ defines an endofunctor of $\hat{\mathcal{O}}$ with biadjoint $V^*\otimes\bullet$

We also have the decomposition according to central characters: for $M\in\hat{\mathcal{O}}$, let M^λ ($\lambda\in\Lambda/W$) to be $\{m\in M \mid \exists\, \varphi_i: Z(\mathfrak{g}) \twoheadrightarrow \hat{R}$, $i=1..k$, w. $\hat\lambda = \pi\circ\varphi_i$ s.t. $(\prod_{i=1}^k \ker\varphi_i)M = 0\}$. Then $M = \bigoplus_\lambda M^\lambda$ (b/c each Verma has an honest central character and M is filt'd by quotients of Verma). We set $\hat{\mathcal{O}}_\lambda = \{M\in\hat{\mathcal{O}} \mid M = M^\lambda\}$

Note that we still have projective functors, translation functors $T_{\lambda\to\mu}: \hat{\mathcal{O}}_\lambda \to \hat{\mathcal{O}}_\mu$ and reflection functors $\mathfrak{R}_i$ defined as in the undeformed case with very similar properties

2.2) *Categorical properties* ($\lambda+\rho$ is domin't)

$\hat{\mathcal{O}}_\lambda$ doesn't have finite length, but it does have fin. many simples: $L(w\cdot\lambda)$, $w\in W/W_{\lambda+\rho}$ and enough projectives: $\hat{\Delta}(\lambda)$ is projective for the same reason as before and every simple is covered by $pr_\lambda(V\otimes\hat{\Delta}(\lambda))$ for a suitable V. Let $\hat{P}(w\cdot\lambda)$ denote the projective cover of $L(w\cdot\lambda)$ (existence & uniqueness is an exercise).

Lem: (a) $\hat{P}(w\cdot\lambda)\twoheadrightarrow\hat{\Delta}(w\cdot\lambda)$ and ker is filt by $\hat{\Delta}(u\cdot\lambda)$ w $u\cdot\lambda>w\cdot\lambda$

(b) $Hom_{\hat{\mathcal{O}}}(\hat{P}_1,\hat{P}_2)$ is a free fin. rk $\hat{R}$-module specializing to $Hom_{\mathcal{O}}(P_1,P_2)$. Here $\hat{P}_i\in\hat{\mathcal{O}}$-proj, $P_i=\hat{P}_i\otimes_{\hat{R}}\mathbb{C}$

(c) $\hat{P}(w\cdot\lambda)\otimes_{\hat{R}}\mathbb{C}=P(w\cdot\lambda)$

Proof: (a) is proved in the same way as in the undeformed case

(b): It's enough to prove this for $\hat{P}_i=V_i\otimes\hat{\Delta}(0)$ - every proj've is a direct summand in here. Then $Hom_{\hat{\mathcal{O}}}(\hat{P}_1,\hat{P}_2)=Hom_{\hat{\mathcal{O}}}(\hat{\Delta}(0),V_2^*\otimes V_1\otimes\hat{\Delta}(0))$ $=pr_\lambda(V_2^*\otimes V_1\otimes\hat{\Delta}(0))_0$. This space is filt'd by $\hat{\Delta}(0)_0$'s w. mult'y equal $\dim pr_\lambda(V_2^*\otimes V_1\otimes\Delta(0))_0$, so it's indeed a free finite rk module. Similarly $pr_\lambda(V_2^*\otimes V_1\otimes\hat{\Delta}(0))_0\otimes_{\hat{R}}\mathbb{C}\xrightarrow{\sim}pr_\lambda(V_2^*\otimes V_1\otimes\Delta(0))_0$. To prove (c) observe that, by (b), $\hat{P}(w\cdot\lambda)\otimes_{\hat{R}}\mathbb{C}$ is indecomposable so it has to coincide w $P(w\cdot\lambda)$ $\square$

We conclude that $\hat{\mathcal{O}}_\lambda\simeq\hat{A}$-mod, where $\hat{A}=End(\bigoplus_{w\in W/W_{\lambda+\rho}}\hat{P}(w\cdot\lambda))^{opp}$ is a free $\hat{R}$-algebra w. $\hat{A}\otimes_{\hat{R}}\mathbb{C}=A$, $\mathcal{O}_\lambda=A$-mod. In this way, $\hat{\mathcal{O}}_\lambda$ is an $\hat{R}$-flat deformation of $\mathcal{O}_\lambda$. In part'r $\hat{\mathcal{O}}_{-\rho}\simeq\hat{R}$-mod

3) *Extended categories* $\hat{\mathcal{O}}$

$\hat{\mathcal{O}}^e_\lambda=\{M\in\mathcal{U}_{\hat{R}}\otimes_{\hat{R}[\mathfrak{h}^*]^{W}}\hat{R}[\mathfrak{h}^*]\text{-mod}\mid M\in\hat{\mathcal{O}}_\lambda$ & $\exists\,\varphi_i:\hat{R}[\mathfrak{h}^*]\twoheadrightarrow\hat{R}$ st comp'n of φ_i w $\hat{R}\twoheadrightarrow\mathbb{C}=\lambda$ & $(\prod_{(i)}\ker\varphi_i)M=0\}$

Relation between $\hat{\mathcal{O}}^e_\lambda$ and $\hat{\mathcal{O}}$ is similar to that between $\tilde{\mathcal{O}}_\lambda$ and $\mathcal{O}_\lambda$ in Mitya's talk. For example, $\hat{\mathcal{O}}^e_\lambda=\hat{\mathcal{O}}_\lambda$ if λ is dominant and $\hat{\mathcal{O}}^e_{-\rho}\xrightarrow{\sim}\hat{R}\otimes_{\hat{R}^W}\hat{R}\text{-mod}$ (the projective in $\hat{\mathcal{O}}^e_{-\rho}$ is $\hat{\Delta}(-\rho)\otimes_{\hat{R}^W}\hat{R}$)

Similarly to Mitya's talk, for $W_{\mu+\rho}\subset W_{\lambda+\rho}$, have the extended translation functor $\tilde{T}_{\lambda\to\mu} : \hat{\mathcal{O}}_\lambda^e \to \hat{\mathcal{O}}_\mu^e$

Then we have the following claims:

Thm (Soergel) 1) $\mathrm{End}_{\hat{\mathcal{O}}_0}(\hat{P}_{min}) = \hat{R} \otimes_{\hat{R}^W} \hat{R}$

2) $\hat{\mathbb{V}} := \mathrm{Hom}_{\hat{\mathcal{O}}_0}(\hat{P}_{min}, \bullet) : \hat{\mathcal{O}}_0 \to \hat{R} \otimes_{\hat{R}^W} \hat{R}\text{-Mod}$ is f[ully]faithful on the projectives

3) $\hat{\mathbb{V}}$ intertwines $\mathcal{P}_i$ w. $\hat{R} \otimes_{\hat{R}^{s_i}} \bullet$

The proofs closely follow those in Mitya's talk ((1)&(2) can also be formally deduced from the corresponding claims in Mitya's talk)

Thm (Soergel) $\hat{\mathbb{V}} : \hat{\mathcal{O}}_0\text{-proj} \xrightarrow{\sim} SBim^\wedge$ (objects are completions of Soergel bimodules & morphisms are homom'sms of $\hat{R} \otimes_{\hat{R}^W} \hat{R}$-modules)

Proof: Recall $\mathrm{Hom}_{R\text{-}R}(B_1, B_2)^{\wedge_0} \xrightarrow{\sim} \mathrm{Hom}_{\hat{R}\text{-}\hat{R}}(B_1^{\wedge_0}, B_2^{\wedge_0})$ for $B_1, B_2 \in SBim$ it follows that if B_i is indec in $SBim$, then $B_i^{\wedge_0}$ is indec in $SBim^\wedge$. Then the proof repeats that in Mitya's talk. $\square$

Proof of Prop 0':
$$\begin{array}{ccc}
\mathrm{Hom}_{\hat{R}\text{-}\hat{R}}(B_1^{\wedge_0}, B_2^{\wedge_0}) \otimes_R \mathbb{C} & \xrightarrow{\sim} & \mathrm{Hom}_R(B_1 \otimes_R \mathbb{C}, B_2 \otimes_R \mathbb{C}) \\
\uparrow \wr & & \uparrow \wr \\
\mathrm{Hom}_{\hat{\mathcal{O}}_R^0}(\hat{P}_1, \hat{P}_2) \otimes_R \mathbb{C} & \xrightarrow[\text{(b) of Lem.}]{\sim} & \mathrm{Hom}_{\mathcal{O}_0}(P_1, P_2)
\end{array}$$

where $\hat{\mathbb{V}}(P_i) = B_i^{\wedge_0}$ $\square$