Algebraic groups and all in characteristic p.

0) Reminder

$\mathbb{F}$ char p field, G alg. group/$\mathbb{F}$

$\rightsquigarrow \mathfrak{g} = Lie(G)$, $\mathcal{U} := \mathcal{U}(\mathfrak{g})$

Last time, we introduced the p (th power) map

$x \mapsto x^{[p]}: \mathfrak{g} \to \mathfrak{g}$ w. following properties:

i) - defining property: under identification $\mathfrak{g} \xrightarrow{\sim} Vect(G)^G$ ($Vect(G)$ $= \mathcal{D}_G$ in Jay's notation), $x^{[p]} := x^p$ as map $\mathbb{F}[G] \to \mathbb{F}[G]$.

ii) - functoriality: if $\Phi: G \to H$ is alg. group homomorphism & $\varphi := d_1\Phi : \mathfrak{g} \to \mathfrak{h}$, then $\varphi(x)^{[p]} = \varphi(x^{[p]})$

Exercise: for $G = GL_n$, have $x^{[p]} = x^p$ as a matrix.

iii) $\Leftarrow$ ii) & Exercise: for $G \subset GL_n$, $x^{[p]} = x^p$ as a matrix.

iv) — · — · — · — · —: $ad(x^{[p]}) = ad(x)^p$

v) easy: $(ax)^{[p]} = a^p x^{[p]}$ $\forall a \in \mathbb{F}$

Fact: in the free algebra $\mathbb{F}\langle x,y\rangle$, the element $(x+y)^p - x^p - y^p$ is a Lie polynomial in x, y. Denote it by $L(x,y)$.

vi) $\Leftarrow$ Fact: $(x+y)^{[p]} = x^{[p]} + y^{[p]} + L(x,y)$.

Definition : A p-Lie algebra over $\mathbb{F}$ is a Lie algebra together w. a p-map $\bullet^{[p]}$ satisfying properties (iv)-(vi)

Example: An associative algebra A together w. $a^{[p]} := a^p$ is a p-Lie algebra.

1) Center & central reductions of $\mathcal{U}(\mathfrak{g})$.

Consider $\iota: \mathfrak{g} \longrightarrow \mathcal{U}$, $\iota(x) = x^p - x^{[p]}$

filtration deg p deg 1

Exercise: Use (iv) & (vi) to show that:

$\iota(x)$ is central

$\iota(x+y) = \iota(x) + \iota(y)$.

$+$ ι is semilinear: $\iota(ax) = a^p \iota(x)$. Assume from now on: $\mathbb{F}$ is perfect. Can twist $\mathbb{F}$-mult'n on $\mathfrak{g}$ by autom $a \mapsto a^{1/p}$ of $\mathbb{F}$, so ι becomes $\mathbb{F}$-linear. Resulting space is denoted by $\mathfrak{g}^{(1)}$ (Frobenius twist).

So have $\mathbb{F}$-linear $\iota: \mathfrak{g}^{(1)} \longrightarrow$ center of $\mathcal{U}$

$$\begin{tikzcd} \mathfrak{g}^{(1)} \arrow[rr] \arrow[dr] & & \text{center of } \mathcal{U} \\ & S(\mathfrak{g}^{(1)}) \arrow[ur, "\iota"'] & \end{tikzcd}$$

Exercise (on PBW): ι is injective & makes $\mathcal{U}$ into free $S(\mathfrak{g}^{(1)})$-module w. basis $x_1^{d_1} \dots x_n^{d_n}$ w. $d_i \in \{0, .. p-1\}$ (here $x_1, .. x_n$ is a basis in $\mathfrak{g}$).

Def'n: $\iota(S(\mathfrak{g}^{(1)}))$ is called the p-center.

<u>Restricted</u> universal enveloping:

$$\mathcal{U}^0(\mathfrak{g}) = \mathcal{U}(\mathfrak{g}) \otimes_{S(\mathfrak{g}^{(1)})} \mathbb{F}_0 = \mathcal{U}(\mathfrak{g}) / (x^p - x^{[p]} \mid x \in \mathfrak{g}).$$

basis: $x_1^{d_1} \dots x_n^{d_n}$, $d_i \in \{0, .. p-1\}$. $S(\mathfrak{g}^{(1)})$-module on $\mathbb{F}$ w. $\mathfrak{g}^{(1)*} \curvearrowright$ by 0.

Universal property: If A is assoc. algebra (hence p-Lie algebra), then any p-Lie algebra homom. $\mathfrak{g} \to A$ uniquely factors through <u>assoc. alg.</u> homom'm $\mathcal{U}^0(\mathfrak{g}) \to A$.

Remark*: Full center: $G \curvearrowright \mathcal{U} \rightsquigarrow$ subalgebra $\mathcal{U}^G \subset \mathcal{U}^{\mathfrak{g}}$, which is the center. $\mathcal{U}^G$ is called <u>Harish-Chandra center</u> $\mathcal{U}^G \xrightarrow{\sim} \mathbb{F}[\mathfrak{h}^*]^{(W,\cdot)}$. Under modest restrictions on p & on G have Veldkamp's thm:

$$\text{center of } \mathcal{U} \xleftarrow{\sim} \mathcal{U}^G \otimes_{S(\mathfrak{g}^{(1)})^G} S(\mathfrak{g}^{(1)})$$

2) Distribution algebra.

Motivation for why we care: care about rat'l rep's of G

Have forgetful functor $Rat_{fd}(G) \longrightarrow \mathcal{U}(\mathfrak{g})\text{-mod}_{fd}$

fin. dim rational reps

But over $\mathbb{F}$ (alg. closed char p field) this functor is far from equivalence. It's neither essentially surjective (one can show we land in $\mathcal{U}^0(\mathfrak{g})\text{-mod}_{fd}$) nor full:

e.g.: $G = \mathbb{G}_m$, rep'n $V \simeq \mathbb{F}$, $t.v = t^p v$, the corresp'g $\mathfrak{g}$-module is trivial.

Goal: replace $\mathcal{U}(\mathfrak{g})$ w. a diff't algebra, $Dist(G)$, w. "forgetful" functor $Rat_{fd}(G) \longrightarrow Dist(G)\text{-mod}_{fd}$ which is "closer" to being an equivalence. In fact, $\mathcal{U}^0(\mathfrak{g}) \hookrightarrow Dist(G)$ & the functor $Rat_{fd}(G) \to Dist(G)\text{-mod}_{fd}$ lifts $Rat_{fd}(G) \longrightarrow \mathcal{U}^0(\mathfrak{g})\text{-mod}_{fd}$.

2.1) Definition of Dist(G).

Setting: R comm've Noetherian ring, G affine group scheme over R (i.e. $R[G]$ is fin. gen'd comm've Hopf algebra)

$\mathfrak{m} = \ker \varepsilon_G = \{f \in R[G] \mid f(1)=0\}$

$V \in R\text{-Mod} \rightsquigarrow V^* = \mathrm{Hom}_R(V,R)$

(care about $R = \mathbb{Z}, \mathbb{Q}, \mathbb{F}$).

Assume $R[G]$ is free over R.

$R[G]^*$ is assoc. algebra w.r.t. Δ^*, where $\Delta: R[G] \to R[G] \otimes R[G]$.

Definition: 1) For $n \geq 0$, define $\mathrm{Dist}_{\leq n}(G)$ as $(R[G]/\mathfrak{m}^n)^* \subset R[G]^*$, the modules of distributions of order $\leq n$.

Note $\mathrm{Dist}_{\leq n}(G) \subset \mathrm{Dist}_{\leq n+1}(G)$

2) $\mathrm{Dist}(G) := \bigcup_n \mathrm{Dist}_{\leq n}(G)$.

Claim: $\mathrm{Dist}(G)$ is a Hopf algebra.

Exercise: $\mathrm{Dist}(G) \subset R[G]^*$ is a subalgebra.

Coproduct on $\mathrm{Dist}(G)$: mult'n $\mu: R[G] \otimes R[G] \to R[G] \rightsquigarrow$

$\mu: R[G]/\mathfrak{m}^n \otimes R[G]/\mathfrak{m}^n \longrightarrow R[G]/\mathfrak{m}^n \rightsquigarrow$

$\mu^*: \mathrm{Dist}_{\leq n}(G) \longrightarrow \mathrm{Dist}_{\leq n}(G) \otimes \mathrm{Dist}_{\leq n}(G)$

$\rightsquigarrow$ coproduct $\mathrm{Dist}(G) \longrightarrow \mathrm{Dist}(G) \otimes \mathrm{Dist}(G)$.

Exercise: Define antipode on $\mathrm{Dist}(G)$ and show it's a Hopf algebra

Exercise: (functoriality) $\varphi: G \to H$ alg. gr'p homom'm
$\rightsquigarrow \varphi^*: k[H] \to k[G] \rightsquigarrow \varphi_*: \mathrm{Dist}(G) \to \mathrm{Dist}(H)$ is a Hopf algebra homom'm.

Exercise: (base change) if k' is k-algebra, then
$\mathrm{Dist}(G_{k'}) = k' \otimes_k \mathrm{Dist}(G_k)$

Connection between $U(\mathfrak{g})$ & $\mathrm{Dist}(G)$
$U(\mathfrak{g})$ = left invariant differential operators on G
$\subseteq \mathrm{End}_k(k[G])$
Define a map $U(\mathfrak{g}) \xrightarrow{\eta} k[G]^*$, $a \in U(\mathfrak{g}) \rightsquigarrow$
$[\eta(a)](f) = (a.f)(1)$, $\operatorname{im} \eta \subset \mathrm{Dist}(G)$ & $\eta: U(\mathfrak{g}) \to \mathrm{Dist}(G)$ respects filtrations. Moreover, η is alg. homom'm.

Facts:
- if k is char 0 field, then $\eta: U(\mathfrak{g}) \xrightarrow{\sim} \mathrm{Dist}(G)$.
- if k is char p field, then η factors through $U^0(\mathfrak{g}) \hookrightarrow \mathrm{Dist}(G)$

2.2) 1-dimensional examples.

- $G = \mathbb{G}_a$, $k[G] = k[t]$, $\Delta(t) = t \otimes 1 + 1 \otimes t$, $\mathfrak{m} = (t)$

For $r \geq 0 \rightsquigarrow \gamma_r \in k[G]^*$: $\gamma_r(t^n) = \delta_{r,n}$ so $\gamma_r \in \mathrm{Dist}_{\leq r}(G)$
So $\gamma_0, \gamma_1, \ldots, \gamma_r, \ldots$ form a basis in $\mathrm{Dist}(G)$
$$\gamma_r * \gamma_s(t^n) = \gamma_r \otimes \gamma_s(\Delta(t^n)) = \gamma_r \otimes \gamma_s\Big(\sum_{i=0}^{n} \binom{n}{i} t^i \otimes t^{n-i}\Big)$$
$$= \begin{cases} \binom{n}{r}, & \text{if } n = r+s \\ 0, & \text{else} \end{cases}$$

so $\gamma_r * \gamma_s = \binom{r+s}{s}\gamma_{r+s}$. $\Rightarrow \gamma_1^n = n!\,\gamma_n$

$\mathrm{Dist}(G_{\mathbb{Z}}) = \mathrm{Span}_{\mathbb{Z}}\left(\frac{\gamma_1^i}{i!} \mid i \geq 0\right) \subset \mathrm{Span}_{\mathbb{Q}}(\gamma_1^i) = \mathrm{Dist}(G_{\mathbb{Q}})$.

infinitely generated.

• $G = \mathbb{G}_m$, $k[G] = k[t^{\pm 1}]$, $m = (t-1)$, $\Delta(t) = t \otimes t$

Define $\beta_r \in k[G]^*$ by $\beta_r((t-1)^n) = \delta_{n,r}$

β_i, $i \geq 0$, form (k-basis) in $\mathrm{Dist}(G)$

$\beta_r(t^n) = \binom{n}{r}$

Exercise: $\forall n \Rightarrow n!\,\beta_n = \beta_1(\beta_1 - 1)\ldots(\beta_1 - (n-1))$ so β_n "=" $\binom{\beta_1}{n}$.

So $\mathrm{Dist}(G_{\mathbb{Z}}) = \mathrm{Span}_{\mathbb{Z}}\left(\binom{\beta_1}{i} \mid i \geq 0\right) \subset \mathbb{Q}[\beta_1] = \mathrm{Dist}(G_{\mathbb{Q}})$.

2.3) Dist(G) for s/simple G.

Assume also G is simply connected, want $\mathrm{Dist}(G_{\mathbb{Z}}) \subset \mathrm{Dist}(G_{\mathbb{Q}}) = \mathcal{U}(\mathfrak{g}_{\mathbb{Q}})$.

Notation: $\Pi \subset \Phi_+$ simple & positive roots

$N^-, T, N \subset G$, max. unipotents & max. torus

$\alpha \in \Phi_+ \rightsquigarrow \mathbb{G}_a^{\pm\alpha} \hookrightarrow N^{\pm}$, $\beta \in \Pi \rightsquigarrow \mathbb{G}_m^{\beta} \hookrightarrow T$

$T = \prod_{\beta \in \Pi} \mathbb{G}_m^{\beta}$ as an alg. group, $N^{\pm} = \prod_{\alpha \in \Phi_+} \mathbb{G}_a^{\pm\alpha}$ as a scheme.

Open Bruhat cell

$$\prod_{\alpha \in \Phi_+} \mathbb{G}_a^{-\alpha} \times \prod_{\beta \in \Pi} \mathbb{G}_m^{\beta} \times \prod_{\alpha \in \Phi_+} \mathbb{G}_a^{\alpha} \xleftarrow{\sim} N^- \times T \times N \subset G \qquad (*)$$

contains 1.

$$\mathbb{G}_a^{\pm\alpha} \hookrightarrow G \rightsquigarrow \mathrm{Dist}(\mathbb{G}^{\pm\alpha}_{a,\mathbb{Z}}) \hookrightarrow \mathrm{Dist}(G_{\mathbb{Z}})$$
$$\gamma_1 \longmapsto e_{\pm\alpha}$$

$$\mathbb{G}_m^{\beta} \hookrightarrow G \rightsquigarrow \mathrm{Dist}(\mathbb{G}^{\beta}_{m,\mathbb{Z}}) \hookrightarrow \mathrm{Dist}(G_{\mathbb{Z}})$$
$$\beta_1 \longmapsto \beta^\vee$$

(*) $\rightsquigarrow$ tensor product (over $\mathbb{Z}$) decomp'n of $\mathrm{Dist}(G_{\mathbb{Z}}) \rightsquigarrow$

Theorem: $\mathrm{Dist}(G_{\mathbb{Z}}) \subset U(\mathfrak{g}_{\mathbb{Q}})$ has following additive basis:

some order

$$\prod_{\alpha\in\Phi_+} \frac{e_{-\alpha}^{k_\alpha}}{k_\alpha!} \prod_{\beta\in\Pi} \binom{\beta}{m_\beta} \prod_{\alpha\in\Phi_+} \frac{e_\alpha^{n_\alpha}}{n_\alpha!}$$

where $k_\alpha, n_\alpha, m_\beta \in \mathbb{Z}_{\geqslant 0}$

Notation: $e_\alpha^{(n)} = \frac{e_\alpha^n}{n!}$ (divided power)

3) Frobenius.

3.1) Frobenius homomorphism:
$\mathbb{F}$ perfect char p field, A fin. gen'd comm've $\mathbb{F}$-algebra $\rightsquigarrow X = \mathrm{Spec}(A)$

Basic observation: $f \mapsto f^p : A \to A$, ring endomorphism. Can make it $\mathbb{F}$-linear if we twist $\mathbb{F}$-mult'n on source by $a \mapsto a^{1/p}$ ($a\in\mathbb{F}$). Denote resulting algebra by $A^{(1)}$. So $f \mapsto f^p : A^{(1)} \longrightarrow A$ is an $\mathbb{F}$-algebra homomorphism.
$\longleftrightarrow$ $Fr: X \longrightarrow X^{(1)}$ ($Fr^*(f) = f^p$)

Exercise: if A is defined over $\mathbb{F}_p$, then $A^{(1)} \xrightarrow{\sim} A$ isom'c as $\mathbb{F}$-algebras.

Suppose A is Hopf algebra. Then $f \mapsto f^p$ is Hopf algebra homom'm. Let $G = Spec(A)$ - alg'c group. Then

$Fr: G \to G^{(1)}$ is an alg. group homom'm.

Example: $G = GL_n \Rightarrow G^{(1)} = GL_n$; $Fr: GL_n \to GL_n$

$Fr((a_{ij})) = (a_{ij}^p)$.

3.2) Fr vs distribution algebra.

$Fr: G \to G^{(1)} \rightsquigarrow Fr_*: Dist(G) \to Dist(G^{(1)})$

Example 1: $G = \mathbb{G}_a (= G^{(1)})$, $Dist(G_{\mathbb{F}}) = Span_{\mathbb{F}}(\gamma_i)$,

w. $\gamma_i(t^n) = \delta_{i,n}$.

$[Fr_*(\gamma_i)](t^n) = \gamma_i(Fr^*(t^n)) = \gamma_i(t^{np})$ so

$$Fr_*(\gamma_i) = \begin{cases} \gamma_{i/p}, & \text{if } i \text{ divisible by } p \\ 0, & \text{else} \end{cases}$$

Example 2: $G = \mathbb{G}_m (= G^{(1)})$ $Dist(G_{\mathbb{F}}) = Span_{\mathbb{F}}(\beta_i)$ w.

$\beta_i((t-1)^n) = \delta_{in}$

Then
$$Fr_*(\beta_i) = \begin{cases} \beta_{i/p} & \text{if } i \text{ is divisible by } p \\ 0 & \text{else.} \end{cases}$$

Example 3: G is semisimple & simply connected

$$\begin{array}{ccc} \mathbb{G}_a^\alpha & \hookrightarrow & G \\ {\scriptstyle Fr}\downarrow & & \downarrow{\scriptstyle Fr} \\ \mathbb{G}_a^{\alpha(1)} & \hookrightarrow & G^{(1)} \\ \wr\wr & & \wr\wr \\ \mathbb{G}_a^\alpha & & G \end{array} \quad \rightsquigarrow \quad \begin{array}{ccc} Dist(\mathbb{G}_a^\alpha) & \longrightarrow & Dist(G) \\ \downarrow{\scriptstyle Fr_*} & & \downarrow{\scriptstyle Fr_*} \\ Dist(\mathbb{G}_a^{\alpha(1)}) & \longrightarrow & Dist(G^{(1)}) \end{array}$$

So $Fr_*\left(\prod_{\alpha\in\Phi_+} e_{-\alpha}^{(k_\alpha)} \prod_{\beta\in\Pi} \binom{\beta^\vee}{m_\beta} \prod_{\alpha\in\Phi_+} e_\alpha^{(n_\alpha)}\right)$

$$= \begin{cases} \prod e_{-\alpha}^{(k_\alpha/p)} \prod \binom{\beta^\vee}{m_\beta/p} \prod e_\alpha^{(n_\alpha/p)} & \text{if } p \text{ divides } \underline{\text{all}}\ k_\alpha, m_\beta, n_\alpha \\ 0 & \text{else.} \end{cases}$$

3.3) Frobenius kernels.

$Fr: G \twoheadrightarrow G^{(1)}$ not isomorphism. It has kernel.

Definition: rth Frobenius kernel $G_r := \ker Fr^r: G \to G^{(r)}$.
(non-reduced group scheme w. single pt. 1)

Example 1: $G = \mathbb{G}_a$, $\mathbb{F}[G] = \mathbb{F}[t]$, $Fr^r: t \mapsto t^{p^r}$

So $G_r = Spec(\mathbb{F}[t]/(t^{p^r}))$; for coproduct $\Delta: \mathbb{F}[t] \to \mathbb{F}[t]^{\otimes 2}$ have $\Delta(t^{p^r}) = t^{p^r}\otimes 1 + 1\otimes t^{p^r}$ so (t^{p^r}) is bialgebra (& Hopf) ideal. So $\mathbb{F}[t]/(t^{p^r})$ is Hopf quotient of $\mathbb{F}[t]$

$Dist(G_r) = \mathbb{F}[G_r]^* \hookrightarrow Dist(G)$; last time defined $\gamma_i \in Dist(G)$ by $\gamma_i(t^n) = \delta_{in}$, these form basis in $Dist(G)$. Then $Dist(G_r) = Span_{\mathbb{F}}(\gamma_0, \dots \gamma_{p^r-1})$.

General case: $\mathbb{F}[G_r]$ is Hopf quotient of $\mathbb{F}[G]$ & $Dist(G_r)$ is a Hopf subalgebra of $Dist(G)$.

Exercise: $G=\mathbb{G}_m$; $\mathbb{F}[G_r]=\mathbb{F}[t,t^{-1}]/(t^{p^r}-1)$ &
$\mathrm{Dist}(G_r)=\mathrm{Span}(\beta_0,\dots,\beta_{p^r-1})$ where β_i : $\beta_i((t-1)^n)=\delta_{in}$.

For G semi-simple (& simply conn'd):

Proposition: 1) As a subalgebra of $\mathrm{Dist}(G)$, $\mathrm{Dist}(G_r)$ is spanned by

$$\prod_{\alpha\in\Phi_+} e_{-\alpha}^{(k_\alpha)} \prod_{\beta\in\Pi} \binom{\beta^\vee}{m_\beta} \prod_{\alpha\in\Phi_+} e_\alpha^{(n_\alpha)}\ ;\ 0\le k_\alpha, m_\beta, n_\alpha < p^r.$$

2)
$$\begin{tikzcd} \mathcal{U}(\mathfrak{g}) \arrow[r] \arrow[d, two heads] & \mathrm{Dist}(G) \\ \mathcal{U}^0(\mathfrak{g}) \arrow[r, "\sim"] & \mathrm{Dist}_1(G) \arrow[u, hook] \end{tikzcd}$$

Not really a proof: 1) for the same price as Theorem above.
2): $\mathcal{U}(\mathfrak{g})\to\mathrm{Dist}(G)$, $e_{\pm\alpha}\in\mathfrak{g}\mapsto e_{\pm\alpha}\in\mathrm{Dist}(G)$, $\beta^\vee\to\beta^\vee$. Subalg. in $\mathrm{Dist}(G)$ gen'd by these elements is $\mathrm{Dist}(G_1)$; see that $e_{\pm\alpha}^p$, $(\beta^\vee)^p-\beta^\vee=0$ in $\mathrm{Dist}(G)$
So $\mathcal{U}(\mathfrak{g})\twoheadrightarrow\mathrm{Dist}(G_1)$ factors through $\mathcal{U}^0(\mathfrak{g})\twoheadrightarrow\mathrm{Dist}(G_1)$ & dimensions are both equal to $p^{\dim\mathfrak{g}}$ so this $\twoheadrightarrow$ is $\xrightarrow{\sim}$. $\square$

4) Rational reps of G vs $\mathrm{Dist}(G)$-modules.

Care about: $\mathrm{Rat}_{fd}(G)$ – finite dimensional rat'l rep's.
$\mathrm{Dist}(G)\text{-mod}_{fd}$: fin. dim. $\mathrm{Dist}(G)$-modules.
What's a connection? Assume $\mathbb{F}$ is alg. closed (of char $p>0$).

4.1) Action of Dist(G) on a rational rep'n.

$M \in Rat(G) =$ comodules over $\mathbb{F}[G]$: $a: M \to \mathbb{F}[G] \otimes M$

For $d \in Dist(G)$ define $d.m$ via: $a(m) = \sum_i a_i \otimes m_i$

$d.m = \sum_i \underbrace{d(a_i)}_{\in \mathbb{F}} m_i$

Exercise: Show this equips M w. $Dist(G)$-module str're (hint: revisit definition of the product on $Dist(G)$).

Example: $G = \mathbb{G}_a$: alg. gr'p homom. $\mathbb{G}_a \to GL(M)$ has form $t \mapsto I + \sum_{i \geq 1} A_i t^i$ (finite sum). Easy to see that γ_i acts by A_i. Then relations in $Dist(G)$: $\gamma_i \gamma_j = \binom{i+j}{i} \gamma_{i+j} \Leftrightarrow$ $t \mapsto I + \sum_{i \geq 1} A_i t^i$ is a group homom'm. So rational rep'n of G is the same thing as fin. dim. $Dist(G)$-module where only finitely many γ_i act by nonzero operators.

Exercise: how about $M = \mathbb{F}^2$ all γ_i's ($i > 0$) act as multiples of $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

4.2) Some collection of results.

$G = \mathbb{G}_m$

Proposition 1: Every rational G-rep $\simeq \bigoplus$ 1-dim'l reps; $\forall$ 1-dim. rep. is given by $t \mapsto t^j$, $j \in \mathbb{Z}$.

What about $Dist(G) = Span_{\mathbb{F}}(\beta_i \mid i \geq 0)$ where $\beta_i = \binom{\beta_1}{i}$. The relations in $Dist(G_{\mathbb{Z}})$ are those of binomial coeff's.

For $x\in\mathbb{Z}_p$ have homom'm $Dist(G_{\mathbb{Z}})\longrightarrow\mathbb{Z}_p$, $\beta_i\mapsto\binom{x}{i}$. $\rightsquigarrow$ $Dist(G)\longrightarrow\mathbb{F}_p\hookrightarrow\mathbb{F}$.

Proposition 2: Every fin. dim'l $Dist(G)$-module $\simeq\bigoplus$ 1 dim'ls & 1-dim'l rep'n comes from an element of $\mathbb{Z}_p$ as above.

Theorem: For G is semisimple & simply conn'd, then our functor $Rat_{f\ell}(G)\longrightarrow Dist(G)\text{-mod}_{f\ell}$ is an equivalence.

4.3) Sketches of proofs.

Sketch of proof of Proposition 1:

The subgroup $\{z\in G \mid z^{\ell}=1 \text{ for some } \ell \text{ coprime to } p\}\subset\mathbb{G}_m$ is Zariski dense and acts by diagonalizable operators on every module. This implies the decomposition into 1-dim'l reps. Classif'n of 1-dimensionals is classical. $\square$

Sketch of proof of Proposition 2:

Note that $\beta_1(\beta_1-1)\dots(\beta_1-(p-1))=p!\beta_p=0$ in $Dist(G)$. So, on every $Dist(G)$-module β_1 acts by a diagonalizable operator w. eigenvalues $0,1,\dots p-1$. Take the direct summand correspon-ding to $i\in\{0,1,\dots p-1\}$. Let $\mathbb{F}_{-i}$ be the 1-dimensional $\mathbb{G}_m$-rep corresponding to the character $t\mapsto t^{-i}$. We can view $\mathbb{F}_{-i}$ as a $Dist(G)$-module. Since $Dist(G)$ is a Hopf algebra, it makes sense to tensor modules. Take the summand, call it M. Replacing M with $M \otimes \mathbb{F}_{-i}$, we can assume β_1 acts by 0.

Recall the Frobenius epimorphism $Fr_*: Dist(G) \twoheadrightarrow Dist(G)$, $\beta_i \mapsto \beta_{i/p}$ if i is divisible by p and to 0 else. The condition that β_1 acts by 0 is equivalent to a module being pulled under Fr_*: say $M = (Fr_*)^*(M_1)$. Then we can argue by induction. □

Sketch of proof of Theorem: We'll show every finite dimensional $Dist(G)$-module, M, comes from a rational G-rep. We have the weight decomposition w.r.t. $Dist(T)$: $M = \bigoplus_{\chi} M_{\chi}$ where χ are p-adic weights. We can talk about highest weight: χ_0 s.t. $\chi_0 + \mu$ is not a weight $\forall$ nonnegative linear combination μ of positive roots. By standard Sl_2 considerations, χ_0 is integral (& dominant). So all weights are integral so the action of $Dist(T)$ comes from an action of T.

Since we have the weight decomposition, the actions of $Dist(\mathbb{G}_\alpha)$ satisfy the finiteness conditions discussed in Example from Section 4.1. So they come from $\mathbb{G}_a$-actions.

On the other hand, $Dist(G)$-action on M restricts to $Dist(G_r)$. This gives a coaction of $\mathbb{F}[G_r]$: $M \to \mathbb{F}[G_r] \otimes M$. The actions are compatible w. inclusions $G_r \hookrightarrow G_{r+1}$ so the

coactions are compatible w. projections $\mathbb{F}[G_{r+1}] \twoheadrightarrow \mathbb{F}[G_r]$
So we get a coaction $M \to \mathbb{F}[G]^{\wedge_1} \otimes M$, where $\mathbb{F}[G]^{\wedge_1} = \varprojlim \mathbb{F}[G_r]$, the completion of $\mathbb{F}[G]^{\wedge_1}$.

Since T and all G_α act on M, we get an action morphism $G^0 \times M \to M$, where $G^0 = N^- \times T \times N$ is the open Bruhat cell. It follows that the coaction map $M \to M \otimes \mathbb{F}[G]^{\wedge_1}$ factors through $M \to M \otimes \mathbb{F}[G^0]$. From here one can deduce that we actually get a coaction map $M \to M \otimes \mathbb{F}[G]$, which is what we need to prove. $\square$