Quantum Groups at a Root of unity

So far studied $U_q(\mathfrak{g})$ a k-algebra with $q \in k^\times$. If q not a root of unity then

$$\mathrm{Rep}(U_q(\mathfrak{g})) \leftrightsquigarrow \mathrm{Rep}(\mathcal{U}(\mathfrak{g})) \text{ in char. } 0.$$

When q is a root of unity then $\mathrm{Rep}(U_q(\mathfrak{g}))$ behaves more like $\mathrm{Rep}(\mathcal{U}(\mathfrak{g}))$ in char. $p > 0$

What is $U_q(\mathfrak{g})$ when q is a root of unity?

De Concini-Kac

Lusztig

Jantzen's Book

- v an ind. / $\mathbb{C}$
- $k = \mathbb{C}(v)$ field of frac. of $\mathbb{C}[v,v^{-1}]$
- $U_k = U_k(\mathfrak{g})$ the k-alg with gens.

$$E_\alpha, F_\alpha, K_\alpha^{\pm} \quad (\alpha \in \Pi).$$

Roughly, De Concini-Kac study the $\mathbb{C}[v,v^{-1}]$-subalgebra of U_k gen. by

$$E_\alpha, F_\alpha, K_\alpha^{\pm}, \frac{K_\alpha - K_\alpha^{-1}}{v - v^{-1}} \quad (\alpha \in \Pi).$$

Lusztig constructs a $\mathbb{Z}[v,v^{-1}]$ alg. by using divided powers of the gens.

$\hookrightarrow$ Analogous to Kostant's $\mathbb{Z}$-form of $\mathcal{U}(\mathfrak{g})$.

First, we recap the char. p story.

Affine Algebraic Groups

k any commutative (unital) ring
k-algebras are commutative and associative.

k-group scheme is a representable functor

$$G = \mathrm{Hom}_{k\text{-alg}}(k[G], -) : \{k\text{-alg}\} \to \{\text{grps}\}$$

We assume it's **algebraic** so we have

$$\underbrace{k[T_1, \dots T_s]}_{\text{poly. ring}} \twoheadrightarrow k[G]$$

Example

$V \cong k^n$ a free k-module. Have a functor $\underline{V}$ s.t.

$$\underline{V}(A) = (V \otimes_k A, +) \cong (A^n, +).$$

$k[\underline{V}] = S(V^*) \cong k[T_1, \dots, T_n]$. Special case:
$\mathbb{G}_a = \underline{k}$ with $k[\mathbb{G}_a] = k[T]$.

The usual group axioms make $k[G]$ into a Hopf algebra.

$$\begin{array}{lll} m_G : G \times G \to G & \leftrightarrow & \Delta_G : k[G] \to k[G] \otimes k[G] \\ 1_G : * \to G & \leftrightarrow & \varepsilon_G : k[G] \to k \\ i_G : G \to G & \leftrightarrow & \sigma_G : k[G] \to k[G] \end{array}$$

Examples

- $G = \mathbb{G}_a \rightsquigarrow k[G] = k[T]$
  $\rightsquigarrow \Delta_G(T) = 1 \otimes T + T \otimes 1$
- $G = \mathbb{G}_m \rightsquigarrow k[G] = k[T, T^{-1}]$
  $\rightsquigarrow \Delta_G(T) = T \otimes T$.

Base Change

Suppose we have a ring hom $k \to k'$. Then we get a k'-group scheme

$$G_{k'} = \mathrm{Hom}(k' \otimes_k k[G], -).$$

Derivations

Let A be a k-alg. and M an A-mod. Say $D : A \to M$ is a k-derivation if it is k-linear and

$$D(fg) = f \cdot D(g) + g \cdot D(f)$$

for all $f, g \in A$. Let $\mathrm{Der}_k(A, M)$ be the set of all k-derivations.

Exercise: $\mathrm{Der}_k(A, A)$ is a Lie algebra with Lie bracket

$$[D, D'] = D \circ D' - D' \circ D.$$

Let $\mathcal{D}_G = \mathrm{Der}_k(k[G], k[G])$. A derivation $D \in \mathcal{D}_G$ is called left invariant if

$$\Delta_G \circ D = (\mathrm{Id} \otimes D) \circ \Delta_G$$

It is easily checked that if $D, D' \in \mathcal{D}_G$ are left invariant then

$$\Delta_G \circ [D, D'] = (\mathrm{Id} \otimes [D, D']) \circ \Delta_G$$

so $[D, D']$ is also left invariant.

Definition
The Lie algebra is the subalgebra $\mathrm{Lie}(G) \subset \mathcal{D}_G$ of left invariant derivations.

Translation Actions
The group $G(k)$ acts on itself via left translations. This determines a hom

$$\lambda : G(k) \longrightarrow GL(k[G])$$

as follows.

Firstly, for each k-alg. A and $f \in k[G]$ we get a hom

$$f_A : G(A) \longrightarrow A$$
$$g \longmapsto g(f).$$

This is natural in A and gives an identification

$$k[G] \cong \mathrm{Nat}(G, \mathbb{A}^1).$$

For each $g \in G(k)$ and $f \in k[G]$ we define $\lambda(g)f \in k[G]$ by setting

$$(\lambda(g)f)_A(x) = f_A(g^{-1}x)$$

for all $x \in G(A)$.

The left invariant derivations satisfy

$$\lambda(g) \circ D = D \circ \lambda(g) \qquad g \in G(k).$$

To see the connection we apply this discussion to $G \times G$ to get

$$k[G] \otimes k[G] \cong \mathrm{Nat}(G \times G, \mathbb{A}^1).$$

In this way we have

$$(\Delta_G f)_A(x, y) = f_A(xy)$$

for all k-alg. A and $x, y \in G(A)$.

Another Interpretation

Let k_ε be the 1-dim $k[G]$-mod defined by

$$f \cdot a = \varepsilon(f)a \qquad f \in k[G],\ a \in k.$$

The following gives another interpretation of $\mathrm{Lie}(G)$.

Proposition

The natural map $\mathrm{Lie}(G) \to \mathrm{Der}_k(k[G], k_\varepsilon)$ given by $D \mapsto \varepsilon_G \circ D$ is an isomorphism.

Proof: If $D \in \mathrm{Lie}(G)$ then

$$D = (\mathrm{Id} \otimes \varepsilon) \circ \Delta \circ D = (\mathrm{Id} \otimes \varepsilon) \circ (\mathrm{Id} \otimes D) \circ \Delta$$
$$= (\mathrm{Id} \otimes \varepsilon \circ D) \circ \Delta$$

so the map is injective. On the other hand one checks that if $D \in \mathrm{Der}_k(k[G], k_\varepsilon)$ then

$$(\mathrm{Id} \otimes D) \circ \Delta \in \mathcal{D}_G$$

is left invariant. $\square$

One More Identification

Let $k[\delta]$ be the dual numbers with $\delta^2 = 0$. The natural map $\phi : k[\delta] \to k$ given by $\phi(\delta) = 0$ gives a group homomorphism

$$\phi^* : G(k[\delta]) \longrightarrow G(k).$$

We want to identify $\mathrm{Lie}(G)$ with $\mathrm{Ker}(\phi^*)$

Now $g \in G(k[\delta])$ is in the kernel iff

$$\varepsilon_G = k[G] \xrightarrow{g} k[\delta] \xrightarrow{\phi} k.$$

Let $\mathfrak{m} = \mathrm{Ker}(\varepsilon_G) = \{ f \in k[G] \mid f(1) = 0 \}$ be the augmentation ideal. Then g factors through

$$k[G]/\mathfrak{m}^2 \cong k1 \oplus \mathfrak{m}/\mathfrak{m}^2$$

because $k[G] = k1 \oplus \mathfrak{m}$. Hence, we have a map $D_g : \mathfrak{m}/\mathfrak{m}^2 \to k$ such that

$$g : (a, b) \longmapsto \varepsilon(a) + D_g(b)\delta$$

Now, for any $a_i, b_i \in k$ we have

$$(a_1+b_1\delta)(a_2+b_2\delta) = a_1a_2 + (a_1b_2+a_2b_2)\delta$$

It follows that we have a map

$$\begin{aligned} \mathrm{Ker}(\phi^*) &\longrightarrow \mathrm{Der}_k(k[G], k_\varepsilon) \\ g &\longmapsto D_g \end{aligned}$$

and this is an isomorphism. Hence

$$\boxed{\mathrm{Lie}(G) \cong \mathrm{Hom}_k(m/m^2, k)}$$

The p-map on Lie(G)

Assume now that $\mathrm{char}(k) = p > 0$. If $D \in \mathcal{D}_G$ is a derivation then by Leibniz's formula we have

$$D^p(ff') = \sum_{i=0}^{p} \binom{p}{i} D^i(f) D^{p-i}(f')$$
$$= f D^p(f') + f' D^p(f).$$

Hence $D^p \in \mathcal{D}_G$. The map $D \mapsto D^p$ on $\mathcal{D}_G$ restricts to a map ${}^{[p]} : \mathrm{Lie}(G) \longrightarrow \mathrm{Lie}(G)$.

Proposition

Any k-alg A is a Lie algebra with Lie bracket $[a,b] = ab - ba$. Moreover,

(i) $\mathrm{ad}(a)^p = \mathrm{ad}(a^p)$ for all $a \in A$.

(ii) $(a+b)^p = a^p + b^p + \sum_{0<r<p} S_r(a,b)$

where $S_r(a,b)$ is a functorial combination of Lie monomials involving a and b.

Exercise: Show (i).

This shows that any k-alg. is a p-restricted or p-Lie algebra, which is a Lie algebra $\mathfrak{g}$ with a map $X \mapsto X^{[p]}$ such that:

(i) $\mathrm{ad}(X^{[p]}) = \mathrm{ad}(X)^p$
(ii) $(cX)^{[p]} = c^p X^{[p]}$ all $c \in k$
(iii) $(X+Y)^{[p]} = X^{[p]} + Y^{[p]} + \sum_{0<r<p} S_r(X,Y)$.

Examples

- $G = \mathbb{G}_a$. As $k[G] = k[T]$ then

$$\mathcal{D}_G = \{ f \, \partial/\partial T \mid f \in k[G] \}$$

The k-span of $\partial/\partial T$ gives $\mathrm{Lie}(G)$. We have $X^{[p]} = 0$ because $(\partial/\partial T)^p = 0$.

- $G = \mathbb{G}_m$. Get $\mathrm{Lie}(G)$ is the k-span of $X = T\,\partial/\partial T$. We have $X^{[p]} = X$.