Quantum Groups at a Root of unity

So far studied $U_q(\mathfrak{g})$ a k-algebra with $q \in k^\times$.
If q not a root of unity then

$$\mathrm{Rep}(U_q(\mathfrak{g})) \leftrightsquigarrow \mathrm{Rep}(\mathcal{U}(\mathfrak{g})) \text{ in char. } 0.$$

When q is a root of unity then $\mathrm{Rep}(U_q(\mathfrak{g}))$ behaves more like $\mathrm{Rep}(\mathcal{U}(\mathfrak{g}))$ in char. $p > 0$

What is $U_q(\mathfrak{g})$ when q is a root of unity?

De Concini - Kac | Lusztig

Jantzen's Book

- v an ind. / $\mathbb{C}$
- $k = \mathbb{C}(v)$ field of frac. of $\mathbb{C}[v, v^{-1}]$
- $U_k = U_k(\mathfrak{g})$ the k-alg with gens.

$$E_\alpha, F_\alpha, K_\alpha^{\pm} \quad (\alpha \in \Pi).$$

Roughly, De Concini - Kac study the $\mathbb{C}[v, v^{-1}]$-subalgebra of U_k gen. by

$$E_\alpha, F_\alpha, K_\alpha^{\pm}, \frac{K_\alpha - K_\alpha^{-1}}{v - v^{-1}} \quad (\alpha \in \Pi).$$

Lusztig constructs a $\mathbb{Z}[v, v^{-1}]$ alg. by using divided powers of the gens.

$\hookrightarrow$ Analogous to Kostant's $\mathbb{Z}$-form of $\mathcal{U}(\mathfrak{g})$.

First, we recap the char. p story.

Affine Algebraic Groups

k any commutative (unital) ring
k-algebras are commutative and associative.

k-group scheme is a representable functor

$$G = \mathrm{Hom}_{k\text{-alg}}(k[G], -) : \{k\text{-alg}\} \to \{\text{grps}\}$$

We assume it's algebraic so we have

$$\underbrace{k[T_1, \dots T_s]}_{\text{poly. ring}} \twoheadrightarrow k[G]$$

Example
$V \cong k^n$ a free k-module. Have a functor $\underline{V}$ s.t.

$$\underline{V}(A) = (V \otimes_k A, +) \cong (A^n, +).$$

$k[\underline{V}] = S(V^*) \cong k[T_1, \dots, T_n]$. Special case:
$\mathbb{G}_a = \underline{k}$ with $k[\mathbb{G}_a] = k[T]$.

The usual group axioms make $k[G]$ into a Hopf algebra.

$$\begin{aligned}
m_G &: G \times G \to G & &\leftrightarrow & \Delta_G &: k[G] \to k[G] \otimes k[G] \\
1_G &: * \to G & &\leftrightarrow & \varepsilon_G &: k[G] \to k \\
i_G &: G \to G & &\leftrightarrow & \sigma_G &: k[G] \to k[G]
\end{aligned}$$

Examples

- $G = \mathbb{G}_a \rightsquigarrow k[G] = k[T]$
  $\rightsquigarrow \Delta_G(T) = 1 \otimes T + T \otimes 1$
- $G = \mathbb{G}_m \rightsquigarrow k[G] = k[T, T^{-1}]$
  $\rightsquigarrow \Delta_G(T) = T \otimes T$.

Base Change
Suppose we have a ring hom $k \to k'$. Then we get a k'-group scheme

$$G_{k'} = \mathrm{Hom}(k' \otimes_k k[G], -).$$

Derivations

Let A be a k-alg. and M an A-mod. Say $D : A \to M$ is a k-derivation if it is k-linear and

$$D(fg) = f \cdot D(g) + g \cdot D(f)$$

for all $f, g \in A$. Let $\mathrm{Der}_k(A, M)$ be the set of all k-derivations.

Exercise: $\mathrm{Der}_k(A, A)$ is a Lie algebra with Lie bracket

$$[D, D'] = D \circ D' - D' \circ D.$$

Let $\mathcal{D}_G = \mathrm{Der}_k(k[G], k[G])$. A derivation $D \in \mathcal{D}_G$ is called left invariant if

$$\Delta_G \circ D = (\mathrm{Id} \otimes D) \circ \Delta_G$$

It is easily checked that if $D, D' \in \mathcal{D}_G$ are left invariant then

$$\Delta_G \circ [D, D'] = (\mathrm{Id} \otimes [D, D']) \circ \Delta_G$$

so $[D, D']$ is also left invariant.

Definition
The Lie algebra is the subalgebra $\mathrm{Lie}(G) \subset \mathcal{D}_G$ of left invariant derivations.

Translation Actions

The group $G(k)$ acts on itself via left translations. This determines a hom

$$\lambda : G(k) \to GL(k[G])$$

as follows.

Firstly, for each k-alg. A and $f \in k[G]$ we get a hom

$$f_A : G(A) \longrightarrow A$$
$$g \longmapsto g(f).$$

This is natural in A and gives an identification

$$k[G] \cong \mathrm{Nat}(G, \mathbb{A}^1).$$

For each $g \in G(k)$ and $f \in k[G]$ we define $\lambda(g) f \in k[G]$ by setting

$$(\lambda(g) f)_A(x) = f_A(g^{-1} x)$$

for all $x \in G(A)$.

The left invariant derivations satisfy

$$\lambda(g) \circ D = D \circ \lambda(g) \qquad g \in G(k).$$

To see the connection we apply this discussion to $G \times G$ to get

$$k[G] \otimes k[G] \cong \mathrm{Nat}(G \times G, \mathbb{A}^1).$$

In this way we have

$$(\Delta_G f)_A(x, y) = f_A(xy)$$

for all k-alg. A and $x, y \in G(A)$.

Another Interpretation

Let k_ε be the 1-dim $k[G]$-mod defined by

$$f \cdot a = \varepsilon(f) a \qquad f \in k[G],\ a \in k.$$

The following gives another interpretation of $\mathrm{Lie}(G)$.

Proposition

The natural map $\mathrm{Lie}(G) \to \mathrm{Der}_k(k[G], k_\varepsilon)$ given by $D \mapsto \varepsilon_G \circ D$ is an isomorphism.

Proof: If $D \in \mathrm{Lie}(G)$ then

$$D = (\mathrm{Id} \otimes \varepsilon) \circ \Delta \circ D = (\mathrm{Id} \otimes \varepsilon) \circ (\mathrm{Id} \otimes D) \circ \Delta$$
$$= (\mathrm{Id} \otimes \varepsilon \circ D) \circ \Delta$$

so the map is injective. On the other hand one checks that if $D \in \mathrm{Der}_k(k[G], k_\varepsilon)$ then

$$(\mathrm{Id} \otimes D) \circ \Delta \in \mathcal{D}_G$$

is left invariant. $\square$

One More Identification

Let $k[\delta]$ be the dual numbers with $\delta^2 = 0$. The natural map $\phi : k[\delta] \to k$ given by $\phi(\delta) = 0$ gives a group homomorphism

$$\phi^* : G(k[\delta]) \to G(k).$$

We want to identify $\mathrm{Lie}(G)$ with $\ker(\phi^*)$

Now $g \in G(k[\delta])$ is in the kernel iff

$$\varepsilon_G = k[G] \xrightarrow{g} k[\delta] \xrightarrow{\phi} k.$$

Let $m = \ker(\varepsilon_G) = \{ f \in k[G] \mid f(1) = 0 \}$ be the augmentation ideal. Then g factors through

$$k[G]/m^2 \cong k1 \oplus m/m^2$$

because $k[G] = k1 \oplus m$. Hence, we have a map $D_g : m/m^2 \to k$ such that

$$g : (a, b) \mapsto \varepsilon(a) + D_g(b)\delta$$

Now, for any $a_i, b_i \in k$ we have

$$(a_1+b_1\delta)(a_2+b_2\delta) = a_1a_2 + (a_1b_2+a_2b_2)\delta$$

It follows that we have a map

$$\begin{aligned} \mathrm{Ker}(\phi^*) &\longrightarrow \mathrm{Der}_k(k[G], k_\varepsilon) \\ g &\longmapsto D_g \end{aligned}$$

and this is an isomorphism. Hence

$$\boxed{\mathrm{Lie}(G) \cong \mathrm{Hom}_k(m/m^2, k)}$$

The p-map on Lie(G)

Assume now that $\mathrm{char}(k) = p > 0$. If $D \in \mathcal{D}_G$ is a derivation then by Leibniz's formula we have

$$\begin{aligned} D^p(ff') &= \sum_{i=0}^{p} \binom{p}{i} D^i(f) D^{p-i}(f') \\ &= f D^p(f') + f' D^p(f). \end{aligned}$$

Hence $D^p \in \mathcal{D}_G$. The map $D \mapsto D^p$ on $\mathcal{D}_G$ restricts to a map ${}^{[p]} : \mathrm{Lie}(G) \longrightarrow \mathrm{Lie}(G)$.

Proposition

Any k-alg A is a Lie algebra with Lie bracket $[a,b] = ab - ba$. Moreover,

(i) $\mathrm{ad}(a)^p = \mathrm{ad}(a^p)$ for all $a \in A$.

(ii) $(a+b)^p = a^p + b^p + \sum_{0<r<p} S_r(a,b)$

where $S_r(a,b)$ is a functorial combination of Lie monomials involving a and b.

Exercise: Show (i).

This shows that any k-alg. is a p-restricted or p-Lie algebra, which is a Lie algebra $\mathfrak{g}$ with a map $X \mapsto X^{[p]}$ such that:

(i) $\mathrm{ad}(X^{[p]}) = \mathrm{ad}(X)^p$
(ii) $(cX)^{[p]} = c^p X^{[p]}$ all $c \in k$
(iii) $(X+Y)^{[p]} = X^{[p]} + Y^{[p]} + \sum_{0<r<p} S_r(X,Y)$.

Examples

- $G = \mathbb{G}_a$. As $k[G] = k[T]$ then

$$\mathcal{D}_G = \{ f\, \partial/\partial T \mid f \in k[G] \}$$

The k-span of $\partial/\partial T$ gives $\mathrm{Lie}(G)$. We have $X^{[p]} = 0$ because $(\partial/\partial T)^p = 0$.

- $G = \mathbb{G}_m$. Get $\mathrm{Lie}(G)$ is the k-span of $X = T\,\partial/\partial T$. We have $X^{[p]} = X$.

The Enveloping Algebra

Let $\mathfrak{g} = \mathrm{Lie}(G)$ and let $\mathcal{U}(\mathfrak{g})$ be the universal enveloping algebra. We fix an embedding

$$(*) \qquad \mathfrak{g} \hookrightarrow \mathcal{U}(\mathfrak{g})$$

once and for all.

We're interested in the centre $Z = Z(\mathcal{U}(\mathfrak{g}))$ of the universal enveloping algebra. Let $\mathrm{ad} : \mathcal{U}(\mathfrak{g}) \to GL(\mathcal{U}(\mathfrak{g}))$ be the usual adjoint rep. of $\mathcal{U}(\mathfrak{g})$. Then $\mathfrak{g}$ acts on $\mathcal{U}(\mathfrak{g})$ via this rep and we have

$$Z = \mathcal{U}(\mathfrak{g})^{\mathfrak{g}}.$$

The Harish-Chandra Centre

As for the translations we have a linear action $\rho : G(k) \to GL(k[G])$ given by conjugation. This defines a linear action $\mathrm{Ad} : G(k) \to GL(\mathfrak{g})$, the adjoint representation. Its differential is $\mathrm{ad} : \mathfrak{g} \to GL(\mathfrak{g})$.

By the universal property Ad extends uniquely to a homomorphism

$$\mathrm{Ad} : G(k) \longrightarrow \mathrm{Aut}_{k\text{-alg}}(\mathcal{U}(\mathfrak{g}))$$

The fixed points $\mathcal{U}(\mathfrak{g})^{G(k)} \subseteq \mathcal{U}(\mathfrak{g})^{\mathfrak{g}}$ give a central subalgebra of the enveloping algebra called the Harish-Chandra centre.

The p-centre

Assume $\mathrm{char}(k) = p > 0$. Recall we have two Lie algebra embeddings

$$\mathcal{U}(\mathfrak{g}) \hookleftarrow \mathfrak{g} \xhookrightarrow{\phi} \mathcal{D}_G$$

Recall that for $X \in \mathfrak{g}$ we have $X^{[p]} \in \mathfrak{g}$ is the unique element satisfying $\phi(X^{[p]}) = \phi(X)^p$. We let X^p be the p^{th} power as an element of $\mathcal{U}(\mathfrak{g})$. In general $X^{[p]} \neq X^p$.

Moreover $X^{[p]} - X^p \in Z(\mathcal{U}(\mathfrak{g}))$ because

$$\mathrm{ad}(X^{[p]} - X^p) = \mathrm{ad}(X)^p - \mathrm{ad}(X)^p = 0$$

We call $Z_p = Z_p(\mathcal{U}(\mathfrak{g})) = \langle X^{[p]} - X^p \mid X \in \mathfrak{g} \rangle$ the p-centre of $\mathcal{U}(\mathfrak{g})$.

Veldkamp's Theorem

We now assume $k = \bar{k}$ is a field with $\mathrm{char}(k) = p > 0$. In addition we assume G is connected reductive. Recall that G satisfies the standard hypotheses if:

(H1) the derived subgroup $G_{der} \leq G$ is simply connected,
(H2) p is good for G,
(H3) there exists a nondegenerate $G(k)$-invariant bilinear form on $\mathfrak{g}$.

Theorem

If G is con. red. and satisfies the standard hypotheses then the following hold:

(i) The natural product map

$$Z_p \otimes_{Z_p^{G(k)}} \mathcal{U}(\mathfrak{g})^{G(k)} \xrightarrow{\sim} Z$$

is an isomorphism.

(ii) Z is a free Z_p-module of rank $p^{\mathrm{rk}(\mathfrak{g})}$, where $\mathrm{rk}(\mathfrak{g})$ is the dimension of a maximal toral subalgebra.

Remark: This is proved in several places. For details see:

- K.A. Brown and I. Gordon, "The ramification of centres: Lie algebras in positive characteristic and quantised enveloping algebras", Math Z. (238), 733–779 (2001)
- R. Tange, "The Zassenhaus Variety of a reductive Lie algebra in positive characteristic", Adv. Math (224), 340–354 (2010).

Also the included references. This result is false when $G = SL_n$ and $p \mid n$. See a paper by A. Braun, J. of Alg. (504), 217–290 (2018).

One can actually give a basis of Z as a Z_p-module. Fix $T \leq G$ a max. torus and let $\mathfrak{t} = \mathrm{Lie}(T)$. It is known that we have an isomorphism

$$\Phi : \mathcal{U}(\mathfrak{g})^{G(k)} \longrightarrow S(\mathfrak{t})^W.$$

If $r = \dim(\mathfrak{t})$ then choose $u_1, \dots, u_r$ s.t. $\Phi(u_1), \dots, \Phi(u_r)$ are alg. independent homogeneous generators. Then

$$(u_1^{a_1} \cdots u_r^{a_r} \mid 0 \leq a_i < p)$$

gives a basis of Z as a free Z_p-module.

The p-enveloping algebra

Continue to assume $p > 0$. The p-restricted structure on $\mathfrak{g}$ remembers information about G. For instance as Lie algebras we have $\mathrm{Lie}(\mathbb{G}_a) \cong \mathrm{Lie}(\mathbb{G}_m)$ but these are not isomorphic as p-restricted Lie algebras.

The p-enveloping algebra of $\mathfrak{g}$ is defined to be:

$$\mathcal{U}^{[p]}(\mathfrak{g}) = \mathcal{U}(\mathfrak{g}) / Z_p.$$

This is universal with respect to homomorphisms $\mathfrak{g} \to A$ of p-Lie algebras, where A is a k-alg. equipped with its natural p-restricted structure.

The algebra $\mathcal{U}^{[p]}(\mathfrak{g})$ has a k-basis given by 1 and the monomials

$$X_{i_1}^{a_1} \cdots X_{i_r}^{a_r}, \quad \text{with } i_1 < \dots < i_r \text{ and } 0 < a_i < p,$$

where $(X_j)_{j \in J}$ is a totally ordered k-basis of $\mathfrak{g}$.

The Distribution Algebra

The Lie algebra is a first order approximation of G. More generally the distribution algebra takes into account higher order approximations of G. One can consider distributions as a generalisation of derivations to more general differential operators.

For a k-module V let $V^* = \mathrm{Hom}_k(V,k)$ be the dual.

As before $\mathfrak{m} = \ker(\varepsilon_G) = \{f \in k[G] \mid f(1) = 0\}$ is the augmentation ideal. The distributions of order $\leq n$ are

$$\begin{aligned} \mathrm{Dist}_n(G) &= \{\mu : k[G] \to k \mid \mu(\mathfrak{m}^{n+1}) = 0\} \\ &\cong \left(k[G]/\mathfrak{m}^{n+1}\right)^* \end{aligned}$$

We call $\mathrm{Dist}(G) = \bigcup_{n \geq 0} \mathrm{Dist}_n(G)$ the distribution algebra of G.

Exercise

Show that $\mathrm{Dist}(G)$ is a subalgebra of $k[G]^*$ equipped with the product

$$\begin{tikzcd} k[G] \otimes k[G] \arrow[r, "\mu \otimes \nu"] & k \otimes k \arrow[d, "\wr"] \\ k[G] \arrow[u, "\Delta"] \arrow[r, dashed, "\mu\nu"] & k \end{tikzcd}$$

Example

$G = \mathbb{G}_a \rightsquigarrow k[G] = k[T]$

$\rightsquigarrow \mathfrak{m} = \langle T \rangle$

$\rightsquigarrow k[G]/\mathfrak{m}^{n+1} = \bigoplus_{i=0}^{n} kS^i$ with $S = T + \mathfrak{m}^{n+1}$

Let $\gamma_r \in k[T]^*$ be such that $\gamma_r(T^s) = \delta_{rs}$ (the Kronecker delta). Then

$$\mathrm{Dist}(G) = \bigoplus_{r \geq 0} k\gamma_r \quad \text{and} \quad \mathrm{Dist}_n(G) = \bigoplus_{r=0}^{n} k\gamma_r .$$

For the product recall that $\Delta(T) = T \otimes 1 + 1 \otimes T$. So,

$\Delta(T^n) = \sum_{r=0}^{n} \binom{n}{r} T^r \otimes T^{n-r}$. It follows that

$$\gamma_n \gamma_m = \binom{n+m}{n} \gamma_{n+m} \Rightarrow \gamma_1^n = n!\,\gamma_n.$$

Assume $k = \mathbb{Z}$. If K is another ring then $K[G_K] \cong K \otimes_{\mathbb{Z}} k[G]$ and

$$\mathrm{Hom}_K(K[G_K], K) \cong K \otimes_{\mathbb{Z}} \mathrm{Hom}(k[G], k).$$

One similarly checks that $\mathrm{Dist}(G_K) \cong K \otimes_{\mathbb{Z}} \mathrm{Dist}(G)$. Now $\mathrm{Dist}(G_{\mathbb{C}}) \cong \mathbb{C}[\gamma_1]$ is a poly. ring and we can identify $\mathrm{Dist}(G)$ with the lattice

$$\mathrm{Span}_{\mathbb{Z}}\{\gamma_1^n / n! \mid n \geq 1\} \subseteq \mathrm{Dist}(G_{\mathbb{C}}).$$

Example

$G = \mathbb{G}_m \rightsquigarrow k[G] = k[T, T^{-1}]$

$\rightsquigarrow \mathfrak{m} = \langle T - 1 \rangle$

$\rightsquigarrow k[G]/\mathfrak{m}^{n+1} = \bigoplus_{i=0}^{n} kS$ with $S = T - 1 + \mathfrak{m}^{n+1}$

There is a unique $\beta_r \in \mathrm{Dist}(G)$ with

$$\beta_r((T-1)^s) = \delta_{rs}$$

Expanding $T^n = ((T-1)+1)^n$ we get $\beta_r(T^n) = \binom{n}{r}$ for all $n \in \mathbb{Z}$ and $r \in \mathbb{N}$.

If $k = \mathbb{C}$ we get

$$\mathrm{Dist}(G) = \bigoplus_{r \geq 0} k\beta_r \quad \text{and} \quad \mathrm{Dist}_n(G) = \bigoplus_{r=0}^{n} k\beta_r.$$

As $\Delta(T) = T \otimes T$ we get that

$$\Delta(T-1) = (T-1) \otimes (T-1) + (T-1) \otimes 1 + 1 \otimes (T-1).$$

From this we can show that

$$\beta_r \beta_s = \sum_{i=0}^{\min\{r,s\}} \frac{(r+s-i)!}{(r-i)!\,(s-i)!\,i!} \beta_{r+s-i}.$$

As a special case $\beta_1 \beta_r = (r+1)\beta_{r+1} + r\beta_r$
so $(\beta_1 - r)\beta_r = (r+1)\beta_{r+1}$. By induction

$$r!\,\beta_r = \beta_1(\beta_1 - 1)\cdots(\beta_1 - r + 1)$$

so $\beta_r = \binom{\beta_1}{r}$. Thus $\mathrm{Dist}(G) \cong k[\beta_1]$ is again a poly. ring.

Now assume $k = \mathbb{Z}$. For any ring K we have $\mathrm{Dist}(G_K) \cong K \otimes_{\mathbb{Z}} \mathrm{Dist}(G)$ and we can identify $\mathrm{Dist}(G)$ with the lattice

$$\mathrm{Span}_{\mathbb{Z}}\left\{ \binom{\beta_1}{n} \,\middle|\, n \geq 0 \right\} \subseteq \mathrm{Dist}(G_{\mathbb{C}}).$$

The Reductive Case

Now let G be a split con. red. alg. $\mathbb{Z}$-group scheme. We will assume k is a field (or more generally an integral domain). The base change G_k is con. red.

Let $T \leq G$ be a split max. torus and $\Phi \subset X(T)$ the roots. To each root $\alpha \in \Phi$ we fix a root hom. $x_\alpha : \mathbb{G}_{a,\mathbb{Z}} \to G$. We have:

- $t x_\alpha(c) t^{-1} = x_\alpha(\alpha(t)c)$ for all $\mathbb{Z}$-alg. A, $t \in T(A)$, and $c \in A$,
- x_α is an iso. onto its image $U_\alpha \leq G$, the functor $U_\alpha(A) = x_\alpha(A)$,
- $\mathrm{Lie}(U_\alpha) = \mathrm{Lie}(G)_\alpha$ the root space of the Lie alg.

For $\alpha \in \Phi$ we let

$$X_\alpha = (dx_\alpha)(1) \in \mathrm{Lie}(G)_\alpha.$$

Choose a basis $\varphi_1, \dots, \varphi_r$ of $\check{X}(T)$ and set

$$H_i = (d\varphi_i)(1) \in \mathrm{Lie}(T).$$

Lemma
We have $(H_i, X_\alpha \mid 1 \leq i \leq r \text{ and } \alpha \in \Phi)$ is a basis of $\mathrm{Lie}(G)$.

Remark

We have $\mathrm{Lie}(G_k) \cong k \otimes_{\mathbb{Z}} \mathrm{Lie}(G)$ so their canonical images give a basis of $\mathrm{Lie}(G_k)$. Moreover, let

$$H_\alpha = (d\check{\alpha})(1).$$

Then $H_\alpha = [X_\alpha, X_{-\alpha}]$. If G is adjoint then the elements H_α, X_α form a Chevalley basis of $\mathrm{Lie}(G_{\mathbb{C}})$.

For the distribution algebra we have

$$\mathrm{Dist}(G_k) \cong k \otimes_{\mathbb{Z}} \mathrm{Dist}(G).$$

The discussion of $\mathbb{G}_a$ shows that the divided powers $X_\alpha^{(n)} = X_\alpha^n / n!$ form a basis of

$$\mathrm{Dist}(U_\alpha) \subset \mathrm{Dist}(U_{\alpha,\mathbb{C}}).$$

Similarly we have all

$$\binom{H_1}{m_1} \cdots \binom{H_r}{m_r}$$

form a basis of

$$\mathrm{Dist}(T) \subset \mathrm{Dist}(T_{\mathbb{C}}).$$

We get the following PBW type result for $\mathrm{Dist}(G_k)$.

Proposition

Fix a system of positive roots $\Phi^+ \subset \Phi$. Then all terms

$$\prod_{\alpha \in \Phi^+} X_\alpha^{(n_\alpha)} \prod_{i=1}^{r} \binom{H_i}{m_i} \prod_{\alpha \in \Phi^+} X_{-\alpha}^{(n_{-\alpha})}$$

form a basis of $\mathrm{Dist}(G_k)$.

Remark

If G is semisimple and simply connected then one gets that $\mathrm{Dist}(G)$ is Kostant's $\mathbb{Z}$-form of $\mathcal{U}(\mathrm{Lie}(G_{\mathbb{C}}))$.

Modules

k any ring and G any k-group scheme.

So far we have looked at linear actions of $G(k)$. Now we discuss G-modules in earnest.

Let V be a k-mod. Recall $\underline{V}$ is the functor $\underline{V}(A) = (V \otimes_k A, +)$. We say V is a *G-module* if we have a natural transformation

$$G \times \underline{V} \longrightarrow \underline{V}$$

such that for each k-alg. A the map $G(A) \times \underline{V}(A) \to \underline{V}(A)$ defines a k-linear action of $G(A)$ on $\underline{V}(A) \cong V \otimes A$.

Exercise: Consider the morphism $G \times G \to G$ such that $G(A) \times G(A) \to G(A)$ is left translations. Using this make $k[G]$ into a G-module.

The G-modules are equivalent to the *comodules* of the Hopf algebra $k[G]$. These are pairs (M, Δ_M) with M a k-module and $\Delta_M : M \to M \otimes k[G]$ a k-linear map s.t.

$$\begin{tikzcd} M \arrow[r, "\Delta_M"] \arrow[d, "\Delta_M"'] \arrow[dr, phantom, "\circlearrowleft"] & M \otimes k[G] \arrow[d, "Id \otimes \Delta_G"] \\ M \otimes k[G] \arrow[r, "\Delta_M \otimes Id"'] & M \otimes k[G] \otimes k[G] \end{tikzcd} \qquad \begin{tikzcd} M \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowleft"] & M \otimes k[G] \arrow[d, "Id \otimes \varepsilon_G"] \\ M \arrow[r, "\sim"] & M \otimes k \end{tikzcd}$$

(associativity) (identity)

A map $\phi : M \to M'$ is a hom of G-modules iff

$$\Delta_{M'} \circ \phi = (\phi \otimes Id) \circ \Delta_M$$

Dist(G)-Modules

If M is a G-module then this becomes a $\mathrm{Dist}(G)$-module by letting $\mu \in \mathrm{Dist}(G)$ act as the map

$$M \xrightarrow{\Delta_M} M \otimes k[G] \xrightarrow{Id \otimes \mu} M \otimes k \xrightarrow{\sim} M$$

It is obvious that we have

$$\mathrm{Hom}_G(M, M') \subset \mathrm{Hom}_{\mathrm{Dist}(G)}(M, M')$$

but this inclusion may be strict in general. Via the embedding $\mathrm{Lie}(G) \hookrightarrow \mathrm{Dist}(G)$ we have the G-module M also becomes a $\mathrm{Lie}(G)$-module.

How much of the representation theory of G does $\mathrm{Dist}(G)$ see?

Example

$G = \mathbb{G}_a$. Recall $\mathrm{Dist}(G) = \bigoplus_{r \geq 0} k\gamma_r$. Assume M is a G-module. For $m \in M$ we have

$$\Delta_M(m) = \sum_{i \geq 0} m_i \otimes T^i \qquad \text{almost all } m_i = 0.$$

Certainly $\gamma_n m = m_n$ so that

$$\Delta_M(m) = \sum_{n \geq 0} (\gamma_n m) \otimes T^n.$$

Hence the $\mathrm{Dist}(G)$-mod structure uniquely determines the G-mod structure in this case.

Not all loc. finite $\mathrm{Dist}(G)$-modules come from G-modules. For example, if $\mathrm{char}(k) = p > 0$ then k^2 becomes a $\mathrm{Dist}(G)$-module by letting

$$\gamma_i m = \begin{cases} em & \text{if } i = p^r \text{ some } r > 0 \\ m & \text{if } i = 0 \\ 0 & \text{otherwise.} \end{cases}$$

where $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

Remark

Similar calculations can be performed in the case $G = \mathbb{G}_m$. Here one gets again that the $\mathrm{Dist}(G)$-mod structure determines the G-mod structure but not all loc. finite $\mathrm{Dist}(G)$-modules come from G-modules.

The Reductive Case

Now assume G is a split con. red. alg. grp, k is a field, and $T \leq G$ is a split max. torus.

We want to know when $\mathrm{Dist}(G)$-modules come from G-modules. We have just remarked that there are loc. finite $\mathrm{Dist}(G)$-modules that don't come from G-modules when $G = \mathbb{G}_m$. The following corrects for this.

Definition

Assume M is a $\mathrm{Dist}(G)$-module then it is also naturally a $\mathrm{Dist}(T)$-module. We say M is a $\mathrm{Dist}(G)$-T-module if it is loc. finite as a $\mathrm{Dist}(G)$-mod and the $\mathrm{Dist}(T)$-mod structure is induced by a T-mod structure.

Theorem

If M is a projective k-module then there is a bijective correspondence between possible G-mod structures and $\mathrm{Dist}(G)$-T-mod structures on M.

Proof (Sketch): Suppose M is a $\mathrm{Dist}(G)$-T-mod. Given $m \in M$ there is a fin. gen k-submodule $M' \subset M$ containing m. We have $M' = \bigoplus_{\lambda \in X(T)} M'_\lambda$ as a T-mod and only fin. many M'_λ are nonzero.

Hence, for each $m \in M$ and $\alpha \in \Phi$ there exists an integer $n_\alpha(m)$ s.t. $X_\alpha^{(n)} m = 0$ for all $n > n_\alpha(m)$. Define a U_α-mod structure on M by setting

$$\Delta_M(m) = \sum_{n \geq 0} X_\alpha^{(n)} m \otimes X^n$$

where $X \in k[U_\alpha]$ is defined by $X(x_\alpha(c)) = c$. This induces the $\mathrm{Dist}(U_\alpha)$-mod structure.

By assumption we have a T-mod structure inducing the $\mathrm{Dist}(T)$-mod structure. Thus we have homs $T \to GL(M)$ and $U_\alpha \to GL(M)$. A generators and rels. argument says we can glue these to get a hom $G \to GL(M)$. This produces the desired G-mod structure. $\square$

Remark

If G is semisimple and simply connected then $\mathrm{Dist}(G)$-T-modules are the same as loc. finite $\mathrm{Dist}(G)$-modules.

The Frobenius

k a perfect field with $\mathrm{char}(k) = p > 0$.

If A is a k-alg. and $m \in \mathbb{Z}$ then $A^{(m)}$ denotes the k-alg. equal to A as a ring but with

$$b \cdot x = b^{p^{-m}} x \text{ for all } b \in k,\ x \in A^{(m)}.$$

__Remark__
This amounts to twisting the embedding $k \to Z(A)$ by the corresponding automorphism of k.

We define $G^{(r)} = \mathrm{Hom}(k[G]^{(r)}, -)$. Note we have a k-alg. hom

$$\begin{aligned} k[G]^{(r)} &\longrightarrow k[G] \\ a &\longmapsto a^{p^r} \end{aligned}$$

The Frobenius $F^r : G \to G^{(r)}$ is the corresponding morphism.

Suppose now that $G = G_{0,k}$ is base changed from an $\mathbb{F}_p$-group scheme G_0. Then we have $k[G] \cong k \otimes_{\mathbb{F}_p} \mathbb{F}_p[G_0]$ and an isomorphism

$$\begin{aligned} k[G]^{(r)} &\longrightarrow k[G] \\ c \otimes f &\longmapsto c^{p^r} \otimes f. \end{aligned}$$

Hence $G^{(r)} \cong G$ and we can view F^r as an end. of G. In this case we have two natural maps

$$\begin{aligned} k[G] &\longrightarrow k[G] \\ c \otimes f &\longmapsto c^p \otimes f \end{aligned} \qquad\qquad \begin{aligned} k[G] &\longrightarrow k[G] \\ c \otimes f &\longmapsto c \otimes f^p \end{aligned}$$

Arithmetic Frobenius   Geometric Frobenius

As an endomorphism of G we have $F = F^1$ is the morphism whose comorphism is the geometric Frobenius.

Frobenius Kernels

Let $G_r = \mathrm{Ker}(F^r)$ be the kernel of $F^r : G \to G^{(r)}$.
Recall the augmentation ideal

$$\mathfrak{m} = \mathrm{Ker}(\varepsilon_G) = \langle f_1, \dots, f_m \rangle$$

is fin. gen. We have

$$I_r = k[G] \cdot (F^r)^*(\mathfrak{m}) = \sum_{i=1}^{m} k[G] f_i^{p^r}$$

is the ideal defining G_r. Note that we have $\mathfrak{m}^{mp^r} \subset I_r \subset \mathfrak{m}^{p^r}$. At the level of the distribution algebra we get

$$\mathrm{Dist}(G) = \bigcup_{r>0} \mathrm{Dist}(G_r)$$

because

$$\mathrm{Dist}(G_r) \cong \{ \mu \in \mathrm{Dist}(G) \mid \mu(I_r) = 0 \}.$$

We have $I_r \subset \mathfrak{m}^2 \subset \mathfrak{m}$ so

$$(\mathfrak{m}/I_r)/(\mathfrak{m}^2/I_r) \cong \mathfrak{m}/\mathfrak{m}^2$$

which implies $\mathrm{Lie}(G_r) = \mathrm{Lie}(G)$.

Now we can choose the generators $f_1, \dots, f_m$ of $\mathfrak{m}$ s.t. $f_1 + \mathfrak{m}^2, \dots, f_m + \mathfrak{m}^2$ gives a k-basis of $\mathfrak{m}/\mathfrak{m}^2$. The images

$$f_1^{n_1} \cdots f_m^{n_m} + J_r \quad \text{with } 0 \le n_i < p^r$$

gen $k[G_r]$ as a k-module so

$$\boxed{\dim_k(k[G_r]) \le p^{r \dim(G)}}$$

and equality holds if G is reduced.

Distribution vs. Enveloping Algebra

k any commutative ring. Recall that

$$\mathrm{Dist}_1(G) \cong \left(k[G]/m^2 \right)^*.$$

As $k[G] = k1 \oplus m$ we have an embedding

$$\mathrm{Lie}(G) \hookrightarrow \mathrm{Dist}_1(G)$$

with the image being those distributions vanishing at 1. Being an associative algebra $\mathrm{Dist}(G)$ has a natural Lie algebra structure and we get an embedding of Lie algebras

$$\mathrm{Lie}(G) \hookrightarrow \mathrm{Dist}(G).$$

By universality this gives a hom. of k-algebras

$$\Psi : \mathcal{U}(\mathrm{Lie}(G)) \longrightarrow \mathrm{Dist}(G).$$

If $\mathrm{char}(k) = 0$ then Ψ is an isomorphism.

Characteristic p

Now assume $\mathrm{char}(k) = p > 0$. We have $\mathrm{Dist}(G)$ has a natural p-restricted structure and the above embedding is an embedding of p-restricted Lie algebras. Hence Ψ factors through

$$\overline{\Psi} : \mathcal{U}^{[p]}(\mathrm{Lie}(G)) \longrightarrow \mathrm{Dist}(G).$$

Proposition

The map $\overline{\Psi}$ is injective. Moreover, if k is a perfect field then the image of $\overline{\Psi}$ is $\mathrm{Dist}(G_1)$.

Proof:

Count dimensions. $\square$

This implies that the representation theory of G_1 is equivalent to the representation theory of $\mathrm{Lie}(G)$ as a p-restricted Lie algebra.