Let $g = \text{Lie}(G)$ and let $U(g)$ be the universal enveloping algebra. We fix an embedding

\[(*) \quad g \hookrightarrow U(g)\]

once and for all.

We're interested in the centre $Z = Z(U(g))$ of the universal enveloping algebra. Let $\text{ad}: U(g) \to \text{GL}(U(g))$ be the usual adjoint rep. of $U(g)$. Then $g$ acts on $U(g)$ via this rep and we have

$$Z = U(g)^g.$$

The Harish-Chandra Centre

As for the translations we have a linear action $\phi: G(k) \to GL(k[G])$ given by conjugation. This defines a linear action $\text{Ad}: G(k) \to \text{GL}(g)$, the adjoint representation. Its differential is $\text{ad}: g \to GL(g)$.

By the universal property $\text{Ad}$ extends uniquely to a homomorphism

$$\text{Ad}: G(k) \to \text{Aut}_k(U(g)).$$

The fixed points $U(g)^{G(k)} \leq U(g)^g$ give a central subalgebra of the universal enveloping algebra called the Harish-Chandra centre.

The $p$-centre

Assume $\text{char}(k) = p > 0$. Recall we have two Lie algebra embeddings

$$U(g) \longleftrightarrow g \hookrightarrow \mathfrak{g}_0.$$

Recall that for $X \in g$ we have $X^{[p]} \in \mathfrak{g}_0$ is the unique element satisfying $\phi(X^{[p]}) = \phi(X)^p$. We let $X^p$ be the $p$th power as an element of $U(g)$. In general $X^{[p]} \neq X^p$. 

The Enveloping Algebra GL6P
Moreover \( X^{p^2} - X^p \in Z(U(g)) \) because
\[
\text{ad}(X^{p^2} - X^p) = \text{ad}(X)^p - \text{ad}(X)^p = 0
\]

We call \( Z_p = Z_p(U(g)) = \langle X^{p^2} - X^p \mid X \in g \rangle \) the \( p \)-centre of \( U(g) \).

**Veldkamp’s Theorem**

We now assume \( k = \overline{k} \) is a field with \( \text{char}(k) = p > 0 \).

In addition, we assume \( G \) is connected reductive. Recall that \( G \) satisfies the standard hypotheses if:

1. \((H1)\) the derived subgroup \( G_{\text{der}} \leq G \) is simply connected,
2. \((H2)\) \( p \) is good for \( G \),
3. \((H3)\) there exists a nondegenerate \( G(k) \)-invariant bilinear form on \( g \).

**Theorem**

If \( G \) is con. red. and satisfies the standard hypotheses then the following hold:

1. **The natural product map**
   \[
   Z_p \otimes U(g)^G(k) \to Z
   \]
   is an isomorphism.
2. **\( Z \) is a free \( Z_p \)-module of rank \( \text{rk}(g) \)** where \( \text{rk}(g) \) is the dimension of a maximal toral subalgebra.

**Remark:** This is proved in several places. For details see:


Also the included references. This result is false when \( G = \text{SL}_n \) and \( n \). See a paper by A. Braun, J. of Alg (504), 217–290 (2018).
One can actually give a basis of \( \mathbb{Z} \) as a \( \mathbb{Z}_p \)-module. Fix \( T \leq G \) a maximal torus and let 
\( T = \text{Lie}(T) \). It is known that we have an isomorphism
\[
\overline{\varphi} : U(g)^{G(k)} \rightarrow S(T)^W.
\]
If \( r = \dim(T) \) then choose \( u_1, \ldots, u_r \) s.t. \( \overline{\varphi}(u_1), \ldots, \overline{\varphi}(u_r) \)
are alg. independent homogeneous generators. Then
\[
(u_1^{a_1} \cdots u_r^{a_r} | 0 \leq a_i < p)
\]
gives a basis of \( \mathbb{Z} \) as a free \( \mathbb{Z}_p \)-module.

The \( p \)-enveloping algebra

Continue to assume \( p > 0 \). The \( p \)-restricted structure
on \( g \) remembers information about \( G \). For instance as
Lie algebras we have \( \text{Lie}(G_0) \cong \text{Lie}(G_0) \) but these
are not isomorphic as \( p \)-restricted Lie algebras.

The \( p \)-enveloping algebra of \( g \) is defined to be:
\[
U_p[g] = U(g)/\mathbb{Z}_p.
\]
This is universal with respect to homomorphisms
\( g \rightarrow A \) of \( p \)-Lie algebras, where \( A \) is a
\( k \)-alg. equipped with its natural \( p \)-restricted
structure.

The algebra \( U_p[g] \) has a \( k \)-basis given
by \( 1 \) and the monomials
\[
X_{i_1}^{a_1} \cdots X_{i_r}^{a_r}, \text{ with } i_1 < \cdots < i_r \text{ and } \alpha a_i < p,
\]
where \( (X_{i_j})_{1 \leq j \leq r} \) is a totally ordered \( k \)-basis
of \( g \).
The Distribution Algebra

The Lie algebra is a first order approximation of \( G \). More generally, the distribution algebra takes into account higher order approximations of \( G \). One can consider distributions as a generalisation of derivations to more general differential operators.

For a \( k \)-module \( V \), let \( V^* = \text{Hom}_k(V,k) \) be the dual.

As before, \( M = \ker(E) = \{ f \in k[G] | f(1) = 0 \} \) is the augmentation ideal. The distributions of order \( n \) are

\[
\text{Dist}_n(G) = \left\{ u : k[G] \to k \mid u(M^{n+1}) = 0 \right\}
\]

\[
\cong \left( k[G]/M^{n+1} \right)^* \]

We call \( \text{Dist}(G) = \bigoplus_{n \geq 0} \text{Dist}_n(G) \) the distribution algebra of \( G \).

Exercise

Show that \( \text{Dist}(G) \) is a subalgebra of \( k[G]^* \) equipped with the product

\[
k[G] \otimes k[G] \xrightarrow{\mu \otimes \mu} k \otimes k
\]

\[
\Delta \quad \mu \quad \nabla
\]

\[
k[G] \longrightarrow \longrightarrow k
\]

Example

Let \( \chi_r \in k[T]^* \) be such that \( \chi_r(T^g) = S_\lambda \) (the Kronecker delta). Then

\[
\text{Dist}(G) = \bigoplus_{r \geq 0} k\chi_r \quad \text{and} \quad \text{Dist}_n(G) = \bigoplus_{r \geq 0} k\chi_r.
\]

For the product recall that \( \Delta(T) = T \otimes 1 + 1 \otimes T \). So,
\[ \Delta(T^n) = \sum_{r=0}^{n} \binom{n}{r} T^r \otimes T^{n-r}. \] It follows that
\[ \chi_n \chi_m = \binom{n+m}{n} \chi_{n+m} \Rightarrow \chi_i = i! \chi_n. \]

Assume \( k = \mathbb{Z} \). If \( K \) is another ring then \( K[G_K] \cong K \otimes_k K[G] \) and
\[ \text{Hom}_k(K[G_K], K) \cong K \otimes_k \text{Hom}(K[G], k). \]

One similarly checks that \( \text{Dist}(G_K) \cong K \otimes_k \text{Dist}(G) \).
Now \( \text{Dist}(G_{\mathbb{C}}) \cong \mathbb{C}[x] \) is a poly. ring and we can identify \( \text{Dist}(G) \) with the lattice
\[ \text{Span}_k \{ \chi_i/i! : i \geq 1 \} \subseteq \text{Dist}(G_{\mathbb{C}}). \]

Example
\[ G = G_m \quad \Rightarrow \quad K[G] = k[T, T^{-1}] \]
\[ \Rightarrow \quad m = \langle T - 1 \rangle \]
\[ \Rightarrow \quad K[G]/m_{m^1} = \bigoplus_{s=0}^{m_1} kS \quad \text{with} \quad S = T - 1 + m_{m^1}. \]

There is a unique \( \beta_r \in \text{Dist}(G) \) with
\[ \beta_r((T-1)^n) = S_{rs} \]
Expanding \( T^n = ((T-1)+1)^n \) we get \( \beta_r(T^n) = \binom{n}{r} \) for all \( n \in \mathbb{N} \) and \( r \in \mathbb{N}. \)

If \( k = \mathbb{C} \) we get
\[ \text{Dist}(G) = \bigoplus_{r=0}^{n} k \beta_r \quad \text{and} \quad \text{Dist}_n(G) = \bigoplus_{r=0}^{n} k \beta_r. \]

As \( \Delta(T) = T \otimes T \) we get that
\[ \Delta(T-1) = (T-1) \otimes (T-1) + (T-1) \otimes 1 + 1 \otimes (T-1). \]

From this we can show that
\[ \beta_r \beta_s = \sum_{i=0}^{\min\{r, s\}} \frac{(r+s-i)!}{(r-i)!(s-i)!} \beta_{r+s-i}. \]
As a special case \( \beta, \beta_r = (r+1)\beta_{r+1} + r\beta_r \)
so \((\beta,
- r)\beta_r = (r+1)\beta_{r+1} \). By induction

\[
\nu! \beta_r = \beta_1 (\beta_r - 1) \cdots (\beta_r - r + 1)
\]

so \( \beta_r = (\frac{1}{r!}) \). Thus \( \text{Dist}(G) \cong k[\beta] \) is again a poly. ring.

Now assume \( k = \mathbb{Z} \). For any ring \( k \) we have \( \text{Dist}(G_k) \cong k \otimes_k \text{Dist}(G) \) and we can identify \( \text{Dist}(G) \) with the lattice

\[
\text{Span}_k \{\frac{1}{n!} (\beta^r) | n \geq 0 \} \subseteq \text{Dist}(G_k).
\]

**The Reductive Case**

Now let \( G \) be a split con. red. alg. \( k \)-group scheme. We will assume \( k \) is a field (or more generally an integral domain). The base change \( G_k \) is con. red.

Let \( T \subseteq G \) be a split max. torus and \( \Phi \subseteq X(T) \)
the roots. To each root \( \alpha \in \Phi \) we fix a root hom. \( x_\alpha : G_{a,\alpha} \to G \). We have:

- \( \beta x_\alpha(c) \cdot t^\alpha = x_\alpha(t)^{\beta(c)} \) for all \( k \)-alg. \( A \), \( t \in T(A) \), and \( c \in A \).
- \( x_\alpha \) is an iso. onto its image \( U_k \subseteq G \), the functor \( U_k(A) = x_\alpha(A) \).
- \( \text{Lie}(U_k) = \text{Lie}(G) \cdot \alpha \) the root space of the Lie alg.

For \( \alpha \in \Phi \) we let

\[
x_\alpha = (d(x_\alpha))(1) \in \text{Lie}(G) \cdot \alpha.
\]

Choose a basis \( \xi_{\alpha_1}, \ldots, \xi_{\alpha_r} \) of \( X(T) \) and set

\[
\xi_i = (d(\xi_{\alpha_i}))(1) \in \text{Lie}(T).
\]

**Lemma**

We have \((\xi_i, x_\alpha | \alpha \in \Phi \) and \( x_\alpha \in \Phi \) is a basis of \( \text{Lie}(G) \).
Remark
We have $\text{Lie}(G_{k})\cong k\mathfrak{g}$, so their canonical images give a basis of $\text{Lie}(G_{k})$. Moreover, let

$$H_{\alpha} = (d\alpha)(1).$$

Then $H_{\alpha} = [X_{\alpha}, X_{-\alpha}]$. If $G$ is adjoint then the elements $\{H_{\alpha}, X_{\alpha}\}$ form a Chevalley basis of $\text{Lie}(G_{0})$.

For the distribution algebra we have

$$\text{Dist}(G_{k}) \cong k\otimes_{k} \text{Dist}(G).$$

The discussion of $G_{0}$ shows that the divided powers $X_{\alpha}^{(n)} = X_{\alpha}^{n}/n!$ form a basis of

$$\text{Dist}(U_{k}) \subset \text{Dist}(U_{k}, C).$$

Similarly we have all

$$(H_{m_{i}}^{(1)}), \ldots, (H_{m_{r}}^{(1)})$$

form a basis of

$$\text{Dist}(T) \subset \text{Dist}(T_{0}).$$

We get the following PBW type result for $\text{Dist}(G_{k})$.

**Proposition**
Fix a system of positive roots $\Pi^{+} C \Pi$. Then all terms

$$\prod_{\alpha \in \Pi^{+}} X_{\alpha}^{(n_{\alpha})} \prod_{i=1}^{r} (H_{m_{i}}^{(1)}) \prod_{\alpha \in \Pi^{+}} X_{-\alpha}^{(n_{\alpha})}$$

form a basis of $\text{Dist}(G_{k})$.

**Remark**
If $G$ is semisimple and simply connected then one gets that $\text{Dist}(G)$ is Kostant’s Z-fam of $U(\text{Lie}(G_{0}))$.