The Lusztig Form of $U_q(\mathfrak{g})$

- v an indeterminate over $\mathbb{Q}$
- $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$ Laurent polynomial ring
- $\mathcal{R} = \mathbb{Q}(v)$ the fraction field of $\mathcal{A}$.
- $U_\mathcal{R} = U_\mathcal{R}(\mathfrak{g})$ the quantum group as in [J] over $\mathcal{R}$.
- $U_\mathcal{A} \subseteq U_\mathcal{R}$ Lusztig's divided power algebra.

Last time

Recall from last time that for a fixed reduced expression $\underline{w_0}$ of w_0 we have an $\mathcal{A}$-basis of $U_\mathcal{A}$ given by

$$\prod_{j=1}^{N} F_{N+1-j}^{(n_j)} \prod_{\alpha\in\Pi} \left(K_\alpha^{\delta_\alpha}\begin{bmatrix} K_\alpha;0 \\ t_\alpha \end{bmatrix}\right) \prod_{j=1}^{N} E_j^{(n_j)}$$

with $n_j, m_j, t_\alpha \geq 0$ and $\delta_\alpha \in \{0,1\}$. Also,

$$U_\mathcal{A}^+ = \mathrm{Span}_\mathcal{A}\left\{\prod_{j=1}^{N} E_j^{(n_j)}\right\} \quad \text{and} \quad U_\mathcal{A}^- = \mathrm{Span}_\mathcal{A}\left\{\prod_{j=1}^{N} F_j^{(m_j)}\right\}.$$

The algebra $U_\mathcal{A}^0$

We now define an $\mathcal{A}$-subalgebra

$$U_\mathcal{A}^0 = \mathrm{Span}_\mathcal{A}\left\{\prod_{\alpha\in\Delta} K_\alpha^{\delta_\alpha}\begin{bmatrix} K_\alpha;0 \\ t_\alpha \end{bmatrix}\right\}$$

using the basis. Thus we have the familiar triangular decomposition $U_\mathcal{A} \cong U_\mathcal{A}^- \otimes U_\mathcal{A}^0 \otimes U_\mathcal{A}^+$.

Verma Modules and Integrable Modules

A $U_\mathcal{R}$-module M is said to be *integrable* if for any $m \in M$ and $\alpha \in \Pi$ there exists $n_0 \geq 1$ such that

$$E_\alpha^{(n)} m = F_\alpha^{(n)} m = 0 \quad \text{for all } n \geq n_0.$$

As in [J] let Λ denote the weight lattice. Recall from Leonardo's talk that for each $\lambda \in \Lambda$ we have a *Verma module* $M(\lambda) = U_\mathcal{R}/J_\lambda$ where

$$J_\lambda = \sum_{\alpha\in\Pi} U_\mathcal{R} E_\alpha + \sum_{\alpha\in\Pi} U_\mathcal{R}(K_\alpha - v^{(\lambda,\alpha)}).$$

The Verma is not integrable but if λ is dominant then $M(\lambda)$ has an integrable quotient $\tilde{L}(\lambda) \twoheadleftarrow M(\lambda)$ with the kernel given by

$$\sum_{\alpha\in\Pi} U_{\mathbb{R}} F_\alpha^{\langle\lambda,\check{\alpha}\rangle+1} + J_\lambda, \text{ where } \langle\lambda,\check{\alpha}\rangle = 2(\lambda,\alpha)/(\alpha,\alpha).$$

We have an isomorphism of $\mathbb{R}$-vector spaces

$$\begin{aligned} U_{\mathbb{R}}^- &\longrightarrow M(\lambda) \twoheadrightarrow \tilde{L}(\lambda) \\ x &\longmapsto x + J_\lambda \end{aligned}$$

We denote by $M(\lambda)_{\mathcal{A}}$ the image of $U_{\mathcal{A}}^-$ under this map. This is a $U_{\mathcal{A}}$-submodule of $M(\lambda)$ (use the commutator formulas).

Likewise, the image $\tilde{L}(\lambda)_{\mathcal{A}}$ of $U_{\mathcal{A}}^-$ under the natural map $U_{\mathbb{R}}^- \to \tilde{L}(\lambda)$ gives a $U_{\mathcal{A}}$-submodule of $\tilde{L}(\lambda)$ (see Prop. 19.3.2 of [LJ]).

The R-Matrix

Recall from Elijah's talk that we have an element

$$\Theta = \sum_{\mu\geq 0} \Theta_\mu \in (U_{\mathbb{R}} \otimes U_{\mathbb{R}})^\wedge$$

contained in a certain completion of $U_{\mathbb{R}} \otimes U_{\mathbb{R}}$ satisfying

$$\Delta(u)\Theta = \Theta\, {}^{\tau}\Delta(u)$$

for all $u \in U_{\mathbb{R}}$.

Here $\tau : U_{\mathbb{R}} \to U_{\mathbb{R}}$ is the involutive anti-automorphism satisfying

$$\tau(E_\alpha) = E_\alpha, \quad \tau(F_\alpha) = F_\alpha, \quad \tau(K_\alpha) = K_\alpha^{-1},$$

for all $\alpha \in \Pi$ and ${}^{\tau}\Delta = (\tau\otimes\tau)\circ\Delta\circ\tau$.

Write $\underline{w}_0 = (s_{\alpha_1}, \dots, s_{\alpha_N})$ and recall that for $1 \leq j \leq N$ we let $\beta_j = w_j(\alpha_j)$, where $w_j = s_{\alpha_1} \cdots s_{\alpha_{j-1}}$, and

$$E_j^{(s)} = T_{w_j}(E_{\alpha_j}^{(s)}) \quad \text{and} \quad F_j^{(s)} = T_{w_j}(F_{\alpha_j}^{(s)}).$$

In Calder's talks we saw (with slightly different notation) that

$$\Theta = \Theta^{[N]} \cdots \Theta^{[2]} \Theta^{[1]}$$

Where

$$\Theta^{[j]} = \sum_{s \geq 0} (-1)^s v_{\alpha_j}^{-s(s-1)/2} (v_{\alpha_j} - v_{\alpha_j}^{-1})^r [r]_{\alpha_j}^! F_j^{(s)} \otimes E_j^{(s)}$$

This element is contained in a completion $(U_{\mathcal{A}} \otimes_{\mathcal{A}} U_{\mathcal{A}})^\wedge$ hence so is Θ. Thus the components Θ_μ are also contained in this completion.

Recall that the R-matrix is a composition $\Theta \circ \tilde{f} \circ P$. To see that this is realisable over the $\mathcal{A}$-form of the algebra it suffices to see that $\tilde{f}$, defined by

$$\tilde{f}(m \otimes m') = f(\lambda, \mu)\, m \otimes m'$$

is. A discussion of this is carried out in 32.1.4 of [L].

Quantum Frobenius for $\mathfrak{sl}_2$

Fix an integer $\ell \geq 1$ and let $\mathcal{A}_\ell$ be the quotient of $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$ by the ideal generated by the ℓ^{th} cyclotomic polynomial. The image of v in $\mathcal{A}_\ell$ under the natural map has order ℓ. For brevity set $U_\ell = \mathcal{A}_\ell \otimes_{\mathcal{A}} U_{\mathcal{A}}$.

The Case of $\mathfrak{sl}_2$

Assume now that $\mathfrak{g} = \mathfrak{sl}_2$. Let us start with an observation about the algebra U_ℓ.

Let ℓ' be the order of v^2. So, $\ell' = \ell$ if ℓ is odd and $\ell' = \ell/2$ if ℓ is even. In $U_{\mathcal{A}}$ we have the identity

$$\left(\prod_{c=1}^{\ell'} (v^c - v^{-c}) \right) \begin{bmatrix} K; 0 \\ \ell' \end{bmatrix} = \prod_{d=0}^{\ell'-1} (K v^{-d} - K^{-1} v^d)$$

In U_ℓ the LHS is 0.

Exercise: In U_ℓ the RHS is

$$v^{(\ell'-1)\ell'/2} (K^{\ell'} - K^{-\ell'}).$$

Note $v^{(\ell'-1)\ell'/2} = 1$ if $\ell = \ell'$ is odd.

From this we see that $K^{2\ell'} = 1$ in U_ℓ.

Specialisations

If $\varepsilon = \pm 1$ then let $U_\varepsilon = U_{\mathcal{A}} \otimes_{\mathcal{A}} \mathbb{Z}_\varepsilon$, where $\mathbb{Z}_\varepsilon$ is the $\mathcal{A}$-module with $v.c = \varepsilon c$ for all $c \in \mathbb{Z}$. Let

$$U_{\varepsilon,\ell} = \mathcal{A}_\ell \otimes_{\mathbb{Z}} U_\varepsilon,$$

which is again an $\mathcal{A}_\ell$-algebra.

Theorem

Assume ℓ is odd and $\mathfrak{g}=\mathfrak{sl}_2$. Then there exists a unique $\mathcal{A}_\ell$-algebra hom. $\mathrm{Fr}: U_\ell \to U_{1,\ell}$ satisfying:

- $\mathrm{Fr}(E^{(s)}) = E^{(s/\ell)}$ if $\ell | s$ and 0 otherwise
- $\mathrm{Fr}(F^{(s)}) = F^{(s/\ell)}$ if $\ell | s$ and 0 otherwise
- $\mathrm{Fr}(K^{\pm}) = K^{\pm}$.

Remark: The image of Fr is contained in the $\mathbb{Z}$-form $U_{\underline{1}} \subseteq U_{1,\ell}$.

The unicity is clear as $E^{(s)}, F^{(s)}, K^{\pm}$ generate the algebra U_ℓ. We consider the existence.

We use a presentation of $U_\ell \cong U_\ell^- \otimes U_\ell^0 \otimes U_\ell^+$ given by Lusztig, where $U_\ell^{\pm} = \mathcal{A}_\ell \otimes_{\mathcal{A}} U_{\mathcal{A}}^{\pm}$ and $U_\ell^0 = \mathcal{A}_\ell \otimes_{\mathcal{A}} U_{\mathcal{A}}^0$.

The definition of Fr on $U_\ell^{\pm}$ is clear. We thus first define Fr on U_ℓ^0 then consider the relations.

Quantum Frobenius on U_ℓ^0

First recall that $U_{\mathcal{A}}^0$ is generated by $K^{\pm 1}$ and $\begin{bmatrix} K;0 \\ t \end{bmatrix}$ with $t \geq 0$. We define Fr on $U_{\mathcal{A}}^0$ by setting

$$\mathrm{Fr}\left(\begin{bmatrix} K;0 \\ t \end{bmatrix}\right) = \begin{cases} \begin{bmatrix} K;0 \\ t/\ell \end{bmatrix} & \text{if } \ell | t \\ 0 & \text{o/w} \end{cases}$$

What is $\mathrm{Fr}\left(\begin{bmatrix} K;c \\ t \end{bmatrix}\right)$ for $c \geq 0$? In $U_{\mathcal{A}}^0$ we have

$$\begin{bmatrix} K;c \\ t \end{bmatrix} = \sum_{0 \leq j \leq t} v^{c(t-j)} \begin{bmatrix} c \\ j \end{bmatrix} K^{-j} \begin{bmatrix} K;0 \\ t-j \end{bmatrix}.$$

Write $t = t_0 + t_1 \ell$ with $0 \leq t_0 < \ell$. Then we have

$$\mathrm{Fr}\left(\begin{bmatrix} K;c \\ t \end{bmatrix}\right) = \sum_{0 \leq a \leq t_1} v^{\ell c(t_1-a)} \begin{bmatrix} c \\ t_0+a\ell \end{bmatrix} K^{-(t_0+a\ell)} \begin{bmatrix} K;0 \\ t_1-a \end{bmatrix}$$

Now $v^\ell = 1$ and $K^\ell = K$ because $K^2 = 1$ in the algebra U_1 and ℓ is odd. Hence, the expression above becomes,

$$Fr\left(\begin{bmatrix} K;c \\ t \end{bmatrix}\right) = \sum_{0\leq a\leq t_1} \begin{bmatrix} c \\ t_0 + a\ell \end{bmatrix} K^{-(t_0+a)} \begin{bmatrix} K;0 \\ t_1 - a \end{bmatrix}$$

Fact 1: If $\ell \nmid t$ and $\ell \mid c$ then $\begin{bmatrix} c \\ t \end{bmatrix} = 0$ in $\mathcal{A}_\ell$ (see Lem 34.1.2 of [L]).

If $\ell \nmid t$ and $\ell \mid c$ then
$$Fr\left(\begin{bmatrix} K;c \\ t \end{bmatrix}\right) = 0$$

We consider now the case where $\ell' \mid t$ and $\ell' \mid c$. So $t = t_1 \ell'$ and $c = c_1 \ell'$.

Fact 2: If $\ell \mid c$ and $\ell \mid t$ then

$$\begin{bmatrix} c \\ t \end{bmatrix} = \begin{bmatrix} c/\ell \\ t/\ell \end{bmatrix}_{v=1} = \binom{c/\ell}{t/\ell}$$

is a usual binomial coefficient in $\mathcal{A}_\ell$.

With this we have

$$Fr\left(\begin{bmatrix} K;c \\ t \end{bmatrix}\right) = \sum_{0\leq a\leq t_1} \begin{bmatrix} c \\ a\ell \end{bmatrix} K^{-a} \begin{bmatrix} K;0 \\ t_1 - a \end{bmatrix}$$
$$= \begin{bmatrix} K;c_1 \\ t_1 \end{bmatrix}$$

using the fact on binomial coefficients. In other words

If $\ell \mid t$ and $\ell \mid c$ then

$$Fr\left(\begin{bmatrix} K;c \\ t \end{bmatrix}\right) = \begin{bmatrix} K;c/\ell \\ t/\ell \end{bmatrix}$$

Relations

The relations for the algebra $U_{\mathcal{A}}$ (as shown by Lusztig):

(i) $E^{(s)}E^{(t)} = \begin{bmatrix} s+t \\ t \end{bmatrix} E^{(s+t)}$ (i') $F^{(s)}F^{(t)} = \begin{bmatrix} s+t \\ t \end{bmatrix} F^{(s+t)}$

(ii) $K^{\pm}E^{(s)} = v^{\pm 2s} E^{(s)} K^{\pm}$ (ii') $K^{\pm}F^{(s)} = v^{\mp 2s} F^{(s)} K^{\pm}$

(iii) $\begin{bmatrix} K;c \\ t \end{bmatrix} E^{(s)} = E^{(s)} \begin{bmatrix} K;c+2s \\ t \end{bmatrix}$

(iii') $\begin{bmatrix} K;c \\ t \end{bmatrix} F^{(s)} = F^{(s)} \begin{bmatrix} K;c-2s \\ t \end{bmatrix}$

(iv) $E^{(s)}F^{(t)} = \sum_{i=0}^{\min\{s,t\}} F^{(t-i)} \begin{bmatrix} K;2i-s-t \\ t \end{bmatrix} E^{(s-i)}$

and a few more relations amongst $K^{\pm}$, $\begin{bmatrix} K;c \\ t \end{bmatrix}$.
Recall $s,t \geq 0$ and $c \in \mathbb{Z}$ but $t > 0$ in (iii) and (iii').

We show that Fr satisfies (iv). That it satisfies (i), (ii), (i'), (ii') is clear.

Relation (iv)

Clearly we have

$$\mathrm{Fr}\left(E^{(s)}F^{(t)}\right) = \begin{cases} E^{(s/\ell)}F^{(t/\ell)} & \text{if } \ell \mid s \text{ and } \ell \mid t \\ 0 & \text{otherwise.} \end{cases}$$

We can apply this to the RHS of (iv) to get that the i^{th} term in the sum is non-zero only when $s \equiv t \equiv i \pmod{\ell}$.

Hence, we can assume $t = t_0 + t_1\ell$, with $0 \leq t_0 < \ell$, and $s = t_0 + s_1\ell$. Then

$$\mathrm{Fr}(\mathrm{RHS}) = \sum_{a=0}^{\min\{s_1,t_1\}} F^{(t_1-a)} \, \mathrm{Fr}\left(\begin{bmatrix} K;(2a-s_1-t_1)\ell \\ t \end{bmatrix}\right) E^{(s_1-a)}$$

By our previous calculations of $\mathrm{Fr}\left(\begin{bmatrix} K;* \\ * \end{bmatrix}\right)$ we get the desired equality.

The case of ℓ even

What happens if we try to run the previous argument when ℓ is even?

The first problem we see is that $K^{\ell}=1$. If $\ell'=\ell/2$ is odd then we again get that $K^{\ell'}=K$ as before. Now, however, we have $v^{\ell'}=-1$. So we should consider the specialisation U_{-1}. In particular, we have the following generalisation of the previous theorem.

Theorem

Assume $\mathfrak{g}=\mathfrak{sl}_2$ and $\varepsilon=\pm 1$ has order ℓ/ℓ'. If ℓ' is odd then there exists a unique A_ℓ-algebra homomorphism $Fr: U_\ell \to U_{\varepsilon,\ell}$ satisfying:

- $Fr(E^{(s)}) = E^{(s/\ell')}$ if $\ell' | s$ and 0 otherwise
- $Fr(F^{(s)}) = F^{(s/\ell')}$ if $\ell' | s$ and 0 otherwise
- $Fr(K^{\pm}) = K^{\pm}$.

We leave it as an exercise to check the previous arguments go through in this case. However, the following generalisations of our facts will be needed.

Fact 1': If $\ell' \nmid t$ and $\ell' | c$ then $\begin{bmatrix} c \\ t \end{bmatrix} = 0$ in A_ℓ.

Fact 2': If $\ell' | c$ and $\ell' | t$ then

$$\begin{bmatrix} c \\ t \end{bmatrix} = \begin{bmatrix} c/\ell' \\ t/\ell' \end{bmatrix}_{v=\varepsilon}$$

as elements of A_ℓ.

If ℓ' is even then we need to do something else to define the quantum Frobenius. To fix the issue that $K^{\ell'}=1$ Lusztig replaces K by an idempotent.

Quantum Frobenius in General

Now let $\mathfrak{g}$ be an semisimple Lie algebra. We assume $\ell \geq 1$ is an integer and $\mathcal{A}_\ell$ is as before.

For $\alpha \in \Pi$ let ℓ'_α be the order of $v_\alpha = v^{2d_\alpha}$. For odd ℓ one has essentially the same result as for $\mathfrak{sl}_2$.

Theorem

Assume ℓ is odd and $\ell'_\alpha = \ell$ for all $\alpha \in \Pi$ (so ℓ is coprime to 3 if $\mathfrak{g}$ has a factor of type G_2). Then there is a unique $\mathcal{A}_\ell$-algebra homomorphism $\mathrm{Fr}: U_\ell \to U_{1,\ell}$ satisfying:

- $\mathrm{Fr}(E_\alpha^{(s)}) = E_\alpha^{(s/\ell)}$ if $\ell | s$ and 0 otherwise
- $\mathrm{Fr}(F_\alpha^{(s)}) = F_\alpha^{(s/\ell)}$ if $\ell | s$ and 0 otherwise
- $\mathrm{Fr}(K_\alpha^{\pm}) = K_\alpha^{\pm}$

for any $\alpha \in \Pi$.

Lusztig's Setup

The construction of the quantum group in [L] is more general than that of [J]. In full generality it involves root data but we will just consider the semisimple case.

Let $\mathbb{Z}\Pi$ be the free module with basis $\Pi \subseteq \Phi$ and $V = \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}\Pi$ a finite dimensional $\mathbb{Q}$-vector space. We may identify Φ with its natural image in V. Recall $(-,-): V \times V \to \mathbb{Q}$ is the unique W-invariant bilinear form with $(\alpha,\alpha) = 2$ for all short roots $\alpha \in \Phi$.

If $\check{V} = \mathrm{Hom}(V, \mathbb{Q})$ then there is a natural perfect pairing $\langle -,- \rangle : \check{V} \times V \to \mathbb{Q}$ given by

$$\langle f, v \rangle = f(v).$$

For each $\alpha \in \Phi$ let $\check{\alpha} = \frac{2(\alpha,-)}{(\alpha,\alpha)} \in \check{V}$ be the corresponding coroot. Let $\check{\Phi} = \{\check{\alpha} \mid \alpha \in \Phi\}$.

Now pick a finitely generated free $\mathbb{Z}$-module $\mathbb{Z}\Phi \subseteq X \subseteq V$ and assume that

$$\check{X} = \{ y \in \check{V} \mid \langle y, x \rangle \in \mathbb{Z} \text{ for all } x \in X \}$$

contains the coroot lattice $\mathbb{Z}\check{\Phi}$.

Exercise: The restriction of $\langle -, - \rangle$ to $\check{X} \times X$ is a perfect pairing of $\mathbb{Z}$-modules. The tuple $(X, \Phi, \check{X}, \check{\Phi})$ is a root datum.

As in 3.1.1 of [L] we define $U_R(\mathfrak{g}, X)$ to be the R-algebra with generators E_α, F_α ($\alpha \in \Pi$) and K_μ ($\mu \in \check{X}$) satisfying the relations:

(i) $K_0 = 1 \quad K_\mu K_{\mu'} = K_{\mu+\mu'}$ for all $\mu, \mu' \in \check{X}$

(ii) $K_\mu E_\beta = v^{\langle \mu, \beta \rangle} E_\beta K_\mu$ for all $\mu \in \check{X}$ and $\beta \in \Pi$

(iii) $K_\mu F_\beta = v^{-\langle \mu, \beta \rangle} F_\beta K_\mu$ for all $\mu \in \check{X}$ and $\beta \in \Pi$

(iv) $E_\alpha F_\beta - F_\beta E_\alpha = \delta_{\alpha\beta} \dfrac{K_{\check{\alpha}^*} - K_{-\check{\alpha}^*}}{v_\alpha - v_\alpha^{-1}}$ for all $\alpha, \beta \in \Pi$

together with a relation involving the form pairing U_R^+ and U_R^-.

Here $\check{\alpha}^* = d_\alpha \check{\alpha}$ and $v_\alpha = v^{d_\alpha}$, where $d_\alpha = \frac{(\alpha,\alpha)}{2}$. Note,

$$\langle \check{\alpha}^*, - \rangle = (\alpha, -).$$

Hence $K_{\check{\alpha}^*} E_\beta = v^{(\alpha,\beta)} E_\beta K_{\check{\alpha}^*}$ for all $\alpha, \beta \in \Pi$.

Remark: Taking $X = \Lambda$ to be the weight lattice we get $\check{X} = \mathbb{Z}\check{\Phi}$ and $U_R(\mathfrak{g}, \Lambda)$ is the simply connected form of the quantum group.

Remark: To recover the setup in [J] we should arrange that $\check{X} = \mathrm{Span}_{\mathbb{Z}}\{ \check{\alpha}^* \mid \alpha \in \Phi \}$. To do this we dualise our starting setup. Namely for $\alpha \in \Phi$ let

$$\alpha^* = 2\alpha/(\alpha,\alpha) = d_\alpha^{-1}\alpha \in V.$$

Then $\Phi^* = \{\alpha^* \mid \alpha \in \Phi\} \subseteq V$ is a crystallographic root system. If $\Lambda^* \subseteq V$ is its weight lattice then taking $X = \Lambda^*$ we get our desired $\check{X}$.

The category $\mathcal{C}$

> From now on $U_k = U_k(\mathfrak{g}, X)$ denotes the k-algebra defined in [L].

In [L] the role of the weight lattice is played by X. Now Lusztig defines a category $\mathcal{C}$ whose objects are (possibly ∞-dimensional) U_k-modules M equipped with a k-vector space decomposition $M = \bigoplus_{\lambda \in \Lambda} M_\lambda$ into weight spaces:

$$M_\lambda = \{m \in M \mid K_\mu m = v^{\langle \mu, \lambda\rangle} m \text{ for all } \mu \in \check{X}\}$$

Thus $\mathcal{C}$ contains "Type I" modules.

As before we get for each $\lambda \in X$ a Verma module $M(\lambda) = U_k / J_\lambda$ where

$$J_\lambda = \sum_{\alpha \in \Pi} U_k E_\alpha + \sum_{\mu \in \check{X}} U_k(K_\mu - v^{\langle \mu, \lambda\rangle})$$

Let $X^+ = \{\lambda \in X \mid \langle \check{\alpha}, \lambda \rangle \geq 0 \text{ for all } \alpha \in \Pi\}$. Then for each $\lambda \in X^+$ we get an integrable quotient $\tilde{L}(\lambda) \twoheadleftarrow M(\lambda)$ with kernel

$$\sum_{\alpha \in \Pi} U_k F_\alpha^{\langle \check{\alpha}, \lambda\rangle + 1} + J_\lambda.$$

Lusztig's Modified Quantum Group

Not all U_k-modules are contained in $\mathcal{C}$. Lusztig introduces a new algebra whose (unital) modules are those of $\mathcal{C}$.

In 23.1 of [L] Lusztig defines a k-algebra $\dot{U}_k$. We defer the details of this construction to [L]. For each $\lambda \in X$ we have an element $1_\lambda \in \dot{U}_k$ such that

$$\mathbb{1}_\lambda \mathbb{1}_{\lambda'} = \delta_{\lambda,\lambda'} \mathbb{1}_\lambda.$$

Hence $\mathbb{1}_\lambda$ is an idempotent and we have

$$\dot{U}_R = \bigoplus_{\lambda \in X} \dot{U}_R \mathbb{1}_\lambda.$$

Recall this algebra is not unital as $\sum_{\lambda\in X} \mathbb{1}_\lambda \notin X$.

Following Lusztig we say a $\dot{U}_R$-module M is unital if for any $m \in M$ we have:

(i) the set $\{\lambda \in X \mid \mathbb{1}_\lambda m \neq 0\}$ is finite,

(ii) $\sum_{\lambda\in X} \mathbb{1}_\lambda m = m$.

The category $\mathcal{C}$ is then identified with the category of unital $\dot{U}_R$-modules.

We note that the algebra $\dot{U}_R$ has a canonical $\mathcal{A}$-form $\dot{U}_\mathcal{A}$ given by Lusztig's canonical basis.

The Quantum Frobenius

Now let $\ell \geq 1$ be an integer, as before, and let ℓ' be the order of v^2 in $\mathcal{A}_\ell$. So $\ell' = \ell$ if ℓ is odd and $\ell' = \ell/2$ if ℓ is even.

> For simplicity we assume Φ is irreducible and $X = \Lambda$

Let $d_\Phi = \max\{d_\alpha \mid \alpha \in \Phi\} \in \{1,2,3\}$ and for $\alpha \in \Phi$ we set $\tilde{d}_\alpha = d_\Phi d_\alpha^{-1}$. Now set

$$\alpha^* = \begin{cases} \tilde{d}_\alpha \alpha & \text{if } d_\Phi \mid \ell' \\ \alpha & \text{if } d_\Phi \nmid \ell' \end{cases}$$

$$\check{\alpha}^* = \begin{cases} \tilde{d}_\alpha^{-1} \check{\alpha} & \text{if } d_\Phi \mid \ell' \\ \check{\alpha} & \text{if } d_\Phi \nmid \ell' \end{cases}$$

The sets $\Phi^* = \{\alpha^* \mid \alpha \in \Phi\} \subseteq V$ and $\check{\Phi}^* = \{\check{\alpha}^* \mid \alpha \in \Phi\} \subseteq \check{V}$ form root systems with Cartan integers

$$\langle \check{\alpha}^*, \beta^* \rangle = \langle \check{\alpha}, \beta \rangle d_\alpha d_\beta^{-1}.$$

Recall that $\Lambda = \bigoplus_{\alpha\in\Pi} \mathbb{Z}\omega_\alpha$ with $\langle \check{\alpha}, \omega_\beta \rangle = \delta_{\alpha\beta}$. We set

$$\omega_\alpha^* = \begin{cases} \tilde{d}_\alpha \omega_\alpha & \text{if } d_\Phi \mid \ell' \\ \omega_\alpha & \text{if } d_\Phi \nmid \ell' . \end{cases}$$

Then $\Lambda^* = \bigoplus_{\alpha\in\Pi} \mathbb{Z}\omega_\alpha^* \subseteq \Lambda$ is the weight lattice of the root system Φ^*.

Remark: When $2 \mid \ell'$ and Φ is of type B_n then Φ^* is of type C_n and vice versa.

We denote by $\mathfrak{g}^*$ the complex semisimple Lie algebra with root system Φ^*.

Assume $\varepsilon = \pm 1$. As before we can form the specialisation $\dot{U}_\varepsilon = \dot{U}_\mathcal{A} \otimes_\mathcal{A} \mathbb{Z}_\varepsilon$ and the $\mathcal{A}_\ell$-algebra

$$\dot{U}_{\varepsilon,\ell} = \mathcal{A}_\ell \otimes_\mathbb{Z} \dot{U}_\varepsilon .$$

The next result appears as Thm 35.1.9 in [L]. That it holds in the generality we state relies on the remarks in 35.5.2 of [L].

Theorem

Let $\varepsilon = \pm 1$ have order ℓ/ℓ'. There is a unique $\mathcal{A}_\ell$-algebra homomorphism $Fr : \dot{U}_\ell(\mathfrak{g}, \Lambda) \to \dot{U}_{\varepsilon,\ell}(\mathfrak{g}^*, \Lambda^*)$ satisfying:

- $Fr(E_\alpha^{(s)} 1_\lambda) = E_\alpha^{(s/\ell'_\alpha)} 1_\lambda$ if $\ell'_\alpha \mid s$ and $\lambda \in \Lambda^*$ and is 0 otherwise.

- $Fr(F_\alpha^{(s)} 1_\lambda) = F_\alpha^{(s/\ell'_\alpha)} 1_\lambda$ if $\ell'_\alpha \mid s$ and $\lambda \in \Lambda^*$ and is 0 otherwise.

Remark: A presentation for the algebra $\dot{U}_\ell$ is given in 31.1.3 of [L]. There one finds analogues of the relations for $U_R(\mathfrak{sl}_2)$ given earlier.