Title : Category $\mathcal{O}$ of affine Lie algebra.

Date : [2021-4-22 15:00 CDT] and May 6.

Goal : Define category $\mathcal{O}$ for $\hat{\mathfrak{g}}_\kappa$

and the duality functor

$$\left(\mathcal{O}^{op} \underset{\sim}{\xrightarrow{\mathbb{D}}} \mathcal{O} \right)$$

Main Ref :

§1. Setup

- $\mathfrak{g}$: simple Lie algebra $(/\mathbb{C})$.
- G : (cores) simply connected Lie group.
- κ : a bilinear invariant form on $\mathfrak{g}$
- κ_{crit} : $\left(\frac{-1}{2}\right) \cdot \kappa_{Killing}$
- $\hat{\mathfrak{g}}_\kappa$:

as vect space, $\hat{\mathfrak{g}}_\kappa \simeq \mathfrak{g} \otimes \mathbb{C}((t)) \oplus \mathbb{C} \cdot c$

Lie bracket : $[\phi \cdot x, \sigma \cdot x']$

$:= \phi\sigma [x, x'] + [\mathrm{Res}_{t=0}\, \sigma d\phi] \cdot c$

(central on c)

RMK $\hat{\mathfrak{g}}_\kappa^+ := \mathfrak{g}[[t]] \oplus \mathbb{C} \cdot c$

Lie subalg of $\hat{\mathfrak{g}}_\kappa$.

RMK All reps discussed today are of level κ, i.e., c acts as 1.

- KL_κ (the category of Kazhdan-Lusztig modules).

Def: the full subcategory of $\hat{g}_\kappa$-modules ← "exercise" KL_κ is abelian.

s.t. ($V \in \mathrm{obj}\, KL_\kappa$)

$\Updownarrow$

(and

1) $t\mathfrak{g}[[t]]$ acts loc. nilp on V ← RMK this means $\forall v \in V$ $\exists n \in \mathbb{N}$ s.t. $\forall tx_1, tx_2, \ldots, tx_n$, $\left(\prod_{i=1}^{n} (tx_i)\right) v = 0$.

2) ($\mathfrak{g} \curvearrowright V$) integrates to ($G \curvearrowright V$))

- $V(N)$

1) $Q_N := \langle g_1 \cdots g_N \mid g_i \in t\mathfrak{g}[t] \rangle_{\mathbb{C}} \subset U(\hat{g}_\kappa)$

2) let V: $\hat{g}_\kappa$-module. ← RMK By the assumption that $t\mathfrak{g}[[t]]$ acts nilp, $V = \bigcup_{N \in \mathbb{N}} V(N)$ if $V \in KL_\kappa$.

$V(N) := \{ v \in V \mid Q_N v = 0 \}$

- Further assumptions

1) $\kappa \neq \kappa_{crit}$

2) $\left(\frac{\kappa}{\kappa_{Killing}} + \frac{1}{2} \right) \notin \mathbb{Q}_{\geq 0}$ ← only for prop 1(b) !

§2. Structure of Generalized Weyl modules.

Define (Sugawara operator)

$$L_0 := \sum_{j>0} \sum_p (t^{-j} c_p)(t^j c_p) + \underbrace{\sum_p c_p c_p}_{\text{Casimir.}}$$

where $\{c_p | p\}$ is an orthonormal basis of $\mathfrak{g}$ with respect to $-2(\kappa - \kappa_{crit})$.

Fact (Basic Rep Theory)

$$[L_0, t^n x] = n t^n x \qquad \forall n \in \mathbb{Z} \ \forall x \in \mathfrak{g}.$$

Define ((Generalized) Weyl modules).

← RMK. the objects of Cat $\mathcal{O}$ can be defined by quotients of Gen. Weyl modules. (coming later)

1.) $\left\{ M^{\kappa} := \mathrm{Ind}_{\hat{\mathfrak{g}}_\kappa^+}^{\hat{\mathfrak{g}}_\kappa} (M) \;\middle|\; \begin{array}{l} M: \text{a finite dim'l } \mathfrak{g}[t]\text{-rep} \\ \text{extended to } \hat{\mathfrak{g}}_\kappa^+\text{-mod} \\ \text{by letting } c \text{ acts as } 1 \end{array} \right\}$

⊂ the set of generalized Weyl modules.

2) A Weyl module is a generalized Weyl module s.t. M is extended trivially from a fin dim'l $\mathfrak{g}$-irrep. ← i.e. $t\mathfrak{g}[t]$ acts as 0.

Prop 1 Any Gen. Weyl mod has a finite filtration w/ quotients being Weyl mods. ← Prop 1(a) in the main ref.

(proof)

- Let V be a gen. Weyl module w/ M the finite dim'l $G[[t]]$-mod s.t. $V = M^{\kappa}$.
- Define $M = M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_\ell = 0$ ← $\ell < \infty$ since $\dim M < \infty$.

  where $M_{i+1} = t\mathfrak{g}[[t]] M_i$. ← recall $t\mathfrak{g}[[t]]$ acts nilp.
- Thus $t\mathfrak{g}[[t]]$ acts trivially on M_i / M_{i+1}, so $\mathfrak{g}$ (thus G) acts on M_i / M_{i+1}.
- Break the G-mod M_i / M_{i+1} down to G-irreps, giving a refinement of filtration

  $M = M_0 \supsetneq M_{01} \supsetneq M_{02} \supsetneq \cdots \supsetneq M_\ell = 0$

  w/ each subquot being a G-irrep. ← (each G-irrep is automatically a $\hat{\mathfrak{g}}_\kappa^+$ rep by letting $t\mathfrak{g}[[t]] \oplus \mathbb{C}$ act trivially.)
- Induce it by $\mathrm{Ind}_{\hat{\mathfrak{g}}_\kappa^+}^{\hat{\mathfrak{g}}_\kappa}$, we get a finite filtration of Weyl modules. ← since $\mathrm{Ind}_{\hat{\mathfrak{g}}_\kappa^+}^{\hat{\mathfrak{g}}_\kappa}$ is exact... (by Δ-decomp)

$\blacksquare$ for prop 1.

Prop$^{2.}$ For a gen. Weyl mod V, the action of L_0 induces a decomp into a countable direct sum of finite dim'l generalized eigenspaces. ← main ref prop 1.d.

← $\{v \mid (L_0-\lambda I)^n v=0$ for some $n\}$

(Proof). • By Prop 1, it's enough to assume V is a Weyl module V_λ^κ.

• In fact, we will show that V_λ^κ splits into a direct sum of (ordinary) L_0-eigenspaces.

• Let $v\in V_\lambda[\lambda]$. By def, $t\mathfrak{g}[t]$ acts trivially on V_λ, so $L(v)=\Omega(v)=\underbrace{\frac{1}{2}\frac{\kappa}{\kappa-\kappa_{crit}}(\lambda,\lambda+2\rho)}_{P_\kappa(\lambda) :=}\,v$

• By the basic fact that $[L_0,t^n x]=nt^n x$ and direct computation,

$$(t^{-a_1}x_1)\cdots(t^{-a_n}x_n)\,v$$

is a L_0-eigenvector w/ eigenvalue being

$$P_\kappa(\lambda)-(a_1+\dots+a_n)$$

• Recall that $V_\lambda^\kappa := \mathrm{Ind}_{\hat{\mathfrak{g}}_\kappa^+}^{\hat{\mathfrak{g}}_\kappa} V_\lambda$, so

$$V_\lambda^\kappa = \mathrm{span}\left\{(t^{-a_1}x_1)\cdots(t^{-a_n}x_n)\,v \;\middle|\; \begin{matrix} x_i\in\mathfrak{g} \\ a_i\in\mathbb{N}_{\geq 0}\end{matrix}\right\}$$

and hence we're done.

$\blacksquare$ for prop^2.