# Rational Representations in Positive Characteristic

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## §0 Introduction and Notations

We give a quick review of classical results in the rational repn theory of algebraic groups in char $p$.

- Induction functor
- (Dual) Weyl modules
- Steinberg Tensor Product Theorem
- Kempf's Vanishing Theorem
- Linkage Principle

Reference


- Ivan's notes Rational Repns. in Positive Char.
- Jantzen's book Repns of Algebraic Groups.
- Joshua Ciappara and Geordie Williamson Lectures on the Geometry and Modular repn. Theory of Algebraic Groups.


Notations.

- $p$ : prime number
- $\mathbb{F}$: algebraically closed field of char $= p$.
- $G$ : connected algebraic group / $\mathbb{F}$ } unless otherwise stated
- $\mathfrak{g} := \mathrm{Lie}(G)$ over $\mathbb{F}$
- $\mathrm{Rep}(G)$ : the category of all rational repns. of $G$.
- $\mathrm{Rep}_{fd}(G)$ : the category of all finite dim repns. of $G$.
- $\Lambda$: weight lattice of $G$
- $\Lambda^+ \subset \Lambda$: dominant weights
- $T \subset B \subset G$ : maximal torus, Borel.
- $W$ : Weyl group
- $U = R_u(B)$ : unipotent radical of $B$.
- $U^-$ : opposite unipotent
- $\alpha_1, \dots, \alpha_r$ : simple roots of $G$.

- $\alpha_0$: negative root s.t. $\alpha_0^\vee$ is minimal
$\Rightarrow \alpha_0, \ldots, \alpha_r$: simple affine roots
- $w_0$: longest element of $W$.

§1. Family of Rational Representations.

§1.1 Weights.

"Repns of $T/\mathbb{F}$ and Repns of $T/\mathbb{C}$ are NOT different"

· <u>Lemma</u> $\forall\, V \in \mathrm{Rep}(T)$, $V$ is completely reducible and irreds are precisely characters.

$\forall\, V \in \mathrm{Rep}(G)$, $V = \bigoplus_{\lambda \in X} V_\lambda$

$V_\lambda = \{ v \in V \mid t \cdot v = \lambda(t) v,\ \forall\, t \in T \}$

Note that $W = N_G(T)/T$, then $V_\chi \simeq V_{w \cdot \chi}$ $\forall\, \chi \in \Lambda$, $w \in W$.

§1.2

$H \leq G$: algebraic subgroup

We have a natural functor

$\mathrm{res}^G_H : \{G\text{-mod}\} \to \{H\text{-mods}\}$.

<u>Goal</u> Find the right adjoint functor

$\mathrm{Ind}^G_H : \{H\text{-mod}\} \to \{G\text{-mod}\}$

<u>Construction</u>

- "geometric approach"

$\forall$ $M \in \mathrm{Rep}_{\Bbbk}(H)$,

$G \times^H M := G \times M / H$ $\quad H \curvearrowright G \times M$ anti-diagonally

$G \times^H M$ becomes a homog. v.b. $/ G/H$.

<u>$\mathrm{Ind}^G_H M := \Gamma(G/H, G \times^H M)$</u> "algebra interpretation" ✓

Consider the $G \times H$-mod structure on $M \otimes \Bbbk[G]$

$(g, h) \cdot (m, f) := (hm, L(g) R(h) f)$, where $L$ and $g$ denote the left and right regular repns.

$\mathrm{Ind}_H^G(M) = (M \otimes \mathbb{k}[G])^H = \{ f \in \mathrm{Mor}(G, M) \mid f(gh) = h^{-1} f(g)\ \forall g \in G, h \in H \}$

<u>Thm</u> (Frobenius Reciprocity)
$\forall M \in \mathrm{Rep}(H)$, $N \in \mathrm{Rep}(G)$,
$\mathrm{Hom}_G(N, \mathrm{Ind}_H^G M) \simeq \mathrm{Hom}_H(\mathrm{Res}_H^G N, M)$.

<u>§1.3</u> Dual Weyl Modules & Weyl Modules

$\forall \lambda \in \Lambda^+$, $\lambda^* := -w_0(\lambda)$, $w_0 \in W$: longest element.

- <u>Def</u> (Dual Weyl module / Weyl module) ①
  - Dual Weyl module
    $M(\lambda) := \mathrm{Ind}_B^G(\mathbb{F}_{-\lambda^*})$
  - Weyl module ②
    $W(\lambda) := M(\lambda^*)^*$

There is another construction using hyperalgebra.
- <u>Def</u> (Kostant's $\mathbb{Z}$-form)
Let $U_{\mathbb{Z}}(\mathfrak{g}) \subset U(\mathfrak{g})$ be the $\mathbb{Z}$-subalgebra generated by
$\{ e_\alpha^{(n)}, f_\alpha^{(n)} \mid \alpha \in \Phi^+, n \in \mathbb{N} \}$

- <u>Def</u> (Hyperalgebra)
  $U_{\mathbb{F}}(\mathfrak{g}) := U_{\mathbb{Z}}(\mathfrak{g}) \otimes_{\mathbb{Z}} \mathbb{F}$.

- <u>Def</u> (Weyl module)
  $W(\lambda) := U_{\mathbb{F}}(\mathfrak{g}) / I$, where $I$ is the ideal generated by
  - $U_{\mathbb{F}}(\mathfrak{n}^+)^\circ$
  - $h - \lambda(h)$, $h \in U_{\mathbb{F}}(\mathfrak{g})$
  - $(f_\alpha)^{(k)}$, $\alpha \in \Phi^+$, $k > \lambda(h_\alpha)$.

Example $G = SL_2$ $G/B = \mathbb{P}^1$, $\Lambda^+ \simeq \mathbb{Z}_{\geq 0}$

$G \times^B \mathbb{F}_{-n} \simeq \mathcal{O}(n)$

$n) = \Gamma(\mathbb{P}^1, \mathcal{O}(n)) = \mathbb{F}[x,y]_n$

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- Lemma $\forall$ $V \in Rep_{k\alpha}(G)$
  a) $Hom_G(V, M(\lambda)) \simeq Hom_B(V, \mathbb{F}_{-\lambda^*})$
  b) $Hom_G(W(\lambda), V) \simeq Hom_B(\mathbb{F}_\lambda, V)$

Proof a) (easy) exercise

b) $Hom_G(W(\lambda), V) \simeq Hom_G(Ind_B^G(\mathbb{F}_{\lambda^*})^*, V)$

Say "by duality" $\simeq Hom_G(V^*, Ind_B^G(\mathbb{F}_{\lambda^*}))$

$\simeq Hom_B(V^*, \mathbb{F}_{\lambda^*})$

$\simeq Hom_B(\mathbb{F}_\lambda, V)$.

Lemma $\lambda \in \Lambda^+$, $Ind_B^G \mathbb{F}_\lambda \neq 0$. Then $\dim M(\lambda)^U = 1$ and $M(\lambda)^U = \mathbb{F}_\lambda$

Proof.

$M(\lambda) = (\mathbb{F}_{-\lambda^*} \otimes \mathbb{F}[G])^B \Rightarrow$

$M(\lambda)^U = \{ f \in (\mathbb{F}_{-\lambda^*} \otimes \mathbb{F}[G])^B \mid f(u_1 t u_2) = \lambda(t) f(1), \forall u_1 \in U^+, t \in T, u_2 \in U$

Bruhat Thm $\Rightarrow$

$U^+ B \hookrightarrow G$ is dense

$\Rightarrow \dim(M(\lambda)^U) \leq 1$.

On the other hand, $M(\lambda)^U \neq 0$ since $U$ is unipotent. Then $\dim M(\lambda)^U) = 1$

Consider the evaluation map

$\varepsilon : M(\lambda) \to \mathbb{F}_\lambda$ $f \mapsto f(1)$.

Then $\varepsilon$ is a $B$-mod map and is clearly inj. on $M(\lambda)^U$. Then $M(\lambda)^U \subseteq M(\lambda) \Rightarrow M(\lambda)^U = M(\lambda)_\lambda$.

- Cor. $\dim Hom_G(W(\lambda), M(\mu)) = \delta_{\lambda,\mu}$.

- Thm. $\forall \lambda \in \Lambda^+$ $\exists!$ simple repn. $L(\lambda) \in Rep(G)$ w/ highest wt.

$\lambda$. Moreover, $L(\lambda)$ is the unique irred submodule of $M(\lambda)$ and the unique quotient of $W(\lambda)$.

Proof ① $soc_G(M(\lambda))$ is simple

like BGG-category $\mathcal{O}$. The proof of this is similar to the situation in category $\mathcal{O}$

② $\forall \lambda \in \Lambda$, $L(\lambda) := soc_G(M(\lambda))$. Then $L(\lambda)$ has the following prop.

$$L(\lambda)^U = L(\lambda)_\lambda, \text{ and } \dim L(\lambda)^U = 1.$$

③ Frob. Reciprocity $\Rightarrow$
$\forall$ simple $G$-mod. $S$, $\exists \lambda \in \Lambda$ s.t.
$Hom_G(S, Ind_B^G F_\lambda) \neq 0$ i.e. $S = soc_G Ind_B^G F_\lambda$. Existence ✓
Uniqueness follows from ②.

□

Recall that
- $\forall \alpha \in \Phi$, $e_\alpha := (dx_\alpha)(1) \in Lie(G_\mathbb{Z})_\alpha$, $x_\alpha : G_{a,\mathbb{Z}} \to G_\mathbb{Z}$
- Choose a basis $\varphi_1, \dots, \varphi_n$ of $Hom(G_{m,\mathbb{Z}}, T_\mathbb{Z})$. $\beta_i := (d\varphi_i)(1) \in Lie(T_\mathbb{Z})$

Thm (c.f. Ivan's Talk) $Dist(G_\mathbb{Z}) \subset U(\mathfrak{g}_\mathbb{C})$ has the following basis
$$\prod_{\alpha \in \Phi^+} e_{-\alpha}^{(k_\alpha)} \prod_{\beta \in \Pi} \binom{\beta}{m_\beta} \prod_{\alpha \in \Phi^+} e_\alpha^{(n_\alpha)}, \text{ where } k_\alpha, n_\alpha, m_\beta \in \mathbb{Z}_{\geq 0}.$$

Def The Weyl module $W(\lambda) := Dist(G_\mathbb{Z} \otimes_\mathbb{Z} F)$ ~~$\langle \dots, h - \lambda(h), (e_{-\alpha})^{(k)} \rangle$~~

~~argumentation ideal~~

$$Dist(G_F) \langle e_\alpha^{(n_\alpha)}, \binom{\beta}{m_\beta} - \binom{\langle \lambda, \beta^\vee \rangle}{m_\beta}, e_{-\alpha}^{(k_\alpha)} \mid m_\beta > 0, k_\alpha > \langle \lambda, \alpha^\vee \rangle \rangle.$$

In this way $Dist(G)/Dist\, I$ is f.d. and has the same univ. prop.
$\Rightarrow$ coincide w/ previous def.

## §2. Steinberg Tensor Product Theorem.

### §2.1 The Frobenius Morphism.

Recall that we defined the Frobenius morphism

$Fr_G^1 : G \to G^{(1)}$,

in the previous lecture.

If $G$ is defined over $\mathbb{F}_p$, then $G^{(1)} \simeq G$. Composing an isomorphism w/ $Fr_G^1$, we get a Frobenius endomorphism.

$Fr_G : G \to G$.

• **Prop.** (i) Let $V$ be an irred $G^{(1)}$-mod. Then the pull back of $V$ along $Fr_G$ is also irred.

(ii) The pullback of $V$ along $Fr_G$ has highest w.t. $p$ times that of $V$. $Fr^* L(\lambda) \simeq L(p\lambda)$.

**Proof.** Rough ideas

① $Fr_G$ is an epimorphism

② On the maximal torus, $Fr_G$ is the "rising to power $p$" map. □

### §2.2 Main Thm.

Assume that $G$ is semisimple and simply connected. This assumption is "harmless" in the sense that

any reductive group $G$ admits a surj. central isogeny w/ finite kernel from $T \times G'$, where

$T := Z(G)^\circ$, $G'$ : semi-simple & simply connected

**Def.** Let

$$\Lambda_1^+ := \{\lambda \in \Lambda^+ \mid \langle \lambda, \alpha_i^\vee \rangle < p, \ \forall\ i = 1, 2, \dots, r\}.$$

Elements in $\Lambda_1^+$ are called the $p$-restricted weights.

For any $\lambda \in \Lambda^+$,

$\lambda = \sum_{i=0}^{m} p^i \lambda_i$ , $\lambda_0, \cdots, \lambda_m \in \Lambda_1^+$. Note here we may use of our assumption that $G$ is s.s. and s.c.

This expression is unique.

· Thm (i) $L(\lambda) \simeq L(\lambda_0) \otimes Fr_G^* L(\lambda_1) \otimes \cdots \otimes Fr_G^{*m} L(\lambda_m)$.

(ii) $G_1 \hookrightarrow G \to GL(L(\lambda_i))$ is irred. Recall that $G_1$ is the first Frob kernel.

§2.3 The Case of $SL_2$

Lemma Let $i = 0, 1, \cdots, p-1$, and $M$ an irred. $G$-mod. Then $L(i) \otimes Fr_G^*(M)$ is irred.

Proof. Irred. $G$-mods $L(0), \ldots, L(p-1)$ are also irred as $\mathfrak{g}$-modules $\Leftrightarrow$ $G_1$-modules

So, every irred $\mathfrak{g}$-submod of $L(i) \otimes Fr_G^*(M)$ takes the form $L(i) \otimes M_0$, $M_0 \subset Fr_G^*(M)$. If $L(i) \otimes M_0$ is also a $G$-submodule, then $M_0 \subset Fr_G^*(M)$ is a $G^{(1)}$-mod $\Rightarrow$ $L(i) \otimes Fr_G^*(M)$ is irred.

Coro (i) For $i = 0, 1, \cdots, p-1$, $j \in \mathbb{Z}_{\geq 0}$, $L(i + pj) \simeq L(i) \otimes Fr_G^*(L(j))$

(ii) $\forall \lambda \in \Lambda^+$, then

$L(\lambda) \simeq L(\lambda_0) \otimes Fr_G^* L(\lambda_1) \otimes \cdots \otimes Fr_G^*(L(\lambda_m))$, where $\lambda_0, \cdots, \lambda_m$ are as in the previous section.

§2.4 Repns of $G_1$

Motivations. Recall in the $SL_2$-case, we relate the $\mathfrak{g}$-structure on a rational repn of $G_1$ to the $G_1$-module structure.

· Prop Repns of $G_1$ is equivalent to that of $\mathrm{Lie}(G_1)$ as a $p$-Lie algebra.

Proof Idea

Recall that $\forall$ $p$-Lie algebra $(\mathfrak{g}, x \mapsto x^{[p]})$, $U^{[p]}(\mathfrak{g}) := U(\mathfrak{g}) / U(\mathfrak{g}) \langle x^p - x^{[p]} \mid x \in \mathfrak{g} \rangle U(\mathfrak{g})$ the restricted enveloping algebra

$$U^{[p]}(\mathrm{Lie}(G)) \simeq \mathrm{Dist}(G_1)$$

$\square$

· **Prop** Repns of $T_1$ is completely reducible and irred. are parametrized by $\Lambda/p\Lambda$.

**Proof Ideas**

Write down basis in $\mathrm{Dist}(G_1)$. Then triangular decomposition allows us to apply the standard highest wt. theory.

$\square$

Consider

$$M_1(\lambda) := \mathrm{Ind}_{B_1}^{G_1} \mathbb{F}_{-\lambda^*}$$

$$W_1(\lambda) := M_1(\lambda^*)^*$$

Then, $M_1(\lambda) = (\mathbb{F}[G_1] \otimes \mathbb{F}_{-\lambda^*})^{B_1}$

$\simeq \mathbb{F}_0[U_1] \otimes (\mathbb{k}[B_1] \otimes \mathbb{F}_{-\lambda^*})^{B_1}$

$\simeq \mathbb{F}[U_1] \otimes \mathbb{F}_{-\lambda^*}$.

$\Rightarrow \dim M_1(\lambda) = p \dim U_1 = p^{\dim U}$

Similarly, $\dim W_1(\lambda) = p^{\dim U}$.

· **Remark** In fact, we can define $M_r(\lambda) := \mathrm{Ind}_{B_r}^{G_r} \mathbb{F}_{\lambda^*}$, and $W_r(\lambda) := M_r(\lambda^*)^*$. In this case, $\dim M_r(\lambda) = \dim W_r(\lambda) = p^{\cancel{\dim U} \; r|\Phi^+|}$.

· **Prop** $\forall \lambda \in \Lambda/p\Lambda$ $\exists!$ simple $L_1(\lambda) \in \mathrm{Rep}(G_1)$ which is the unique irred subrepn. of $M_1(\lambda)$ and unique quotient of $W_1(\lambda)$.

**Example** $M_1((p-1)\rho) = W_1((p-1)\rho) = L_1((p-1)\rho)$. Steinberg repn of $G_1$

In fact, $\lambda \in \Lambda/p^r\Lambda$, we have $L_r((p^r-1)\lambda) = M_r((p^r-1)\lambda)$: Steinberg repn. Irreducible are parametrized by $\Lambda/p^r\Lambda$.

## §2.5 Proof of the Main Theorem

Again, we assume that $G$ is semi-simple & simple connected.

$\forall \lambda \in \Lambda_1^+$, $\lambda = \sum_{i=0}^{m} p^i \lambda_i$.

• **Thm** $L(\lambda) = L(\lambda_0) \otimes L(\lambda_1)^{(Fr^*)} \otimes \cdots \otimes L(\lambda_m)^{(Fr^*)^m}$

**Proof** ① Let $\lambda \in \Lambda_1^+$. Restriction of $L(\lambda)$ to $G_1$ is isom. to $L_1(\lambda)$.

• $\varphi_g : G \to G$ conj. by $g$. Then $\varphi_g^* L \simeq L$ $\forall$ simple $G_1$-mod $L$ and $g \in G$.

(if $A$ is a finite dim algebra w/ a group homomorphism $G \to \mathrm{Aut}(A)$, then $G \curvearrowright \mathrm{Irr}(A)$. If $G$ is connected, then this action is trivial)

• Given a simple $G_1$-mod $L$, get a proj. repn. of $G$ on $L$.

• (Steinberg's result) For semi-simple and simply connected groups, any proj. repn. lifts to a linear repn. This is standard in char 0.

• Verify that the highest wt. $\lambda'$ of the resulting lift is $\lambda$.

② $\forall \lambda \in \Lambda_1$ and $\mu \in \Lambda_1^+$, $L(\lambda + p\mu) \simeq L(\lambda) \otimes L(\mu)^{(1)}$

The proof in the case of $SL_2$ goes through.

## §3. Kempf's Vanishing Theorem.

Question

How to compute the characters of $M(\lambda)$ and $W(\lambda)$?

We know the situation for $SL_2$:

· Thm. Let $G$ be a connected reductive group. The characters of $M(\lambda)$ and $W(\lambda)$ are the same as that of simple module w/ highest w.t. $\lambda$ in char 0.

### §3.1 Kempf's Thm.

The following Thm is the key ingredient in the proof of the main theorem.

· Thm. $\lambda \in \Lambda^+$. Let $\mathcal{O}(\lambda)$ denote the line bundle $G \times^B \mathbb{F}_\lambda$ over $G/B$. Then

$$H^i(G/B, \mathcal{O}(\lambda)) = 0, \quad \forall i > 0.$$

Proof ① May assume that $G$ is semi-simple and simply connected.

( Can find $G_1$: s.s. and s.c., $G_2$: torus s.t. $G_1 \times G_2 \xrightarrow{p} G$ is a surj. central isogeny. $p^{-1}(B) \subset G) = B_1 \times G_2$, $p^{-1}(T \subset G) = T_1 \times G_2$. $T_1 \subset B_1 \subset G_1$.

$\lambda$ defines a character of $T_1 \times G_2 \Rightarrow \lambda$ gives rise to $\lambda_1 \in \Lambda(T_1)$ and $\lambda_2 \in \Lambda(G_2)$. $\langle \lambda, \alpha^\vee \rangle = \langle \lambda_1, \alpha^\vee \rangle \Rightarrow \lambda_1 \in \Lambda(T_1)^+$.

$G/B \simeq G_1/B_1$ and $\mathcal{O}_{G_1/B_1}(\lambda_1) \simeq \mathcal{O}_{G/B}(\lambda)$ ).

② ~~Rat~~ Rep(B) has enough injectives $\Rightarrow$

$R\mathrm{Ind}_B^G : D^+(\mathrm{Rep}(B)) \to D^+(\mathrm{Rep}(G))$, and in fact

$R\mathrm{Ind}_B^G : D^b(\mathrm{Rep}(B)) \to D^b(\mathrm{Rep}(B))$

Geometrically, $R\mathrm{Ind}_B^G$ is the same as taking cohomology of homogeneous vector bundles over $G/B$.

i.e. $R\mathrm{Ind}_B^G(M) \simeq R\Gamma(G/B, G\times^B M)$, $\forall M\in \mathrm{Rep}_{\le\infty}(B)$.

(3) $\forall \lambda\in\Lambda$, $\mathcal{O}_{G/B}(\lambda)$ is ample $\Leftrightarrow$ $\langle\lambda,\alpha^\vee\rangle>0$ $\forall\alpha\in\Pi$.

(4) By Serre's cohomological criterion for ampleness, $\exists\, r\in\mathbb{N}$ s.t.
$(\lambda+\rho)$ is ample, $H^i(G/B, \mathcal{O}(-\rho)\otimes\mathcal{O}(\lambda+\rho)^{p^r}) = 0$ $\forall\ i>0$.

LHS $= H^i(G/B, \mathcal{O}((p^r-1)\rho + p^r\lambda))$

$= R^i\mathrm{Ind}_B^G(\mathbb{F}_{(p^r-1)\rho}\otimes(\mathrm{Fr}_B^*)^r\mathbb{F}_{-\lambda})$.

(5) Prop $\forall\ r\geq 1$, $M\in\mathrm{Rep}(B)$, Idea of proof: transitivity of induction since $\mathrm{Ind}_B^{G_rB}$ is exact as $G_rB/B$ is affine

$R\mathrm{Ind}_B^G(\mathbb{F}_{-(p^r-1)\rho}\otimes(\mathrm{Fr}_B^*)^r M)\simeq L((p^r-1)\rho)\otimes(\mathrm{Fr}_B^*)^r R\mathrm{Ind}_B^G M$.

Proof

$G_r \triangleleft G_rB$, $G_r\triangleleft G$

$G_rB/G_r \simeq B/B_r$. Consider $(\mathrm{Fr}_B^*)^r M$ as a $G_rB/G_r$-module

$R^i\mathrm{Ind}_{G_rB/G_r}^{G/G_r}((\mathrm{Fr}_B^*)^r M)\simeq R^i\mathrm{Ind}_{G_rB}^G((\mathrm{Fr}_B^*)^rM)$.

Recall that

$$\begin{tikzcd} G/G_r & G_rB/G_r \arrow[l, hook'] \\ G \arrow[u, "\wr"] & B \arrow[l, hook'] \arrow[u, "\wr"'] \end{tikzcd}$$

$\Rightarrow R^i\mathrm{Ind}_{G_rB/G_r}^{G/G_r}((\mathrm{Fr}_B^*)^r M)\simeq R^i\mathrm{Ind}_B^G M$ as $G$-modules (via $G\simeq G/G_r$

Translate $G$-mod structure via $G\to G/G_r$, ~~need~~ requires the Frob twist

$(\mathrm{Fr}_G^*)^r(R^i\mathrm{Ind}_B^G M)\simeq R^i\mathrm{Ind}_{G_rB}^G((\mathrm{Fr}_B^*)^rM)$

$\Rightarrow$

$L((p^r-1)\rho)\otimes(\mathrm{Fr}_G^*)^r(R\mathrm{Ind}_B^G M)$

$\simeq L((p^r-1)\rho)\otimes R^i\mathrm{Ind}_{G_rB}^G((\mathrm{Fr}_B^*)^rM)$

$\simeq R^i\mathrm{Ind}_{G_rB}^G(L((p^r-1)\rho)\otimes(\mathrm{Fr}_B^*)^rM)$ $(*)$

- $L((p^r-1)\rho)\simeq \mathrm{ind}_B^{G_rB}(\mathbb{F}_{(p^r-1)\rho})$ (Exercise)

$\Rightarrow R^i \mathrm{Ind}_{G_rB}^{G} (L((p^r-1)\rho) \otimes (Fr_B^*)^r M)$
$\simeq R^i \mathrm{Ind}_{G_rB}^{G} \circ \mathrm{Ind}_{B}^{G_rB} (\mathbb{k}_{(p^r-1)\rho} \otimes (Fr_B^*)^r M)$ (*)

Note that $G_rB/B \simeq G_r/B_r \triangleleft U_r$ is affine $\Rightarrow \mathrm{Ind}_B^{G_rB}$ is exact
Hence (*) $= R^i \mathrm{Ind}_{G_rB}^{G} \circ R\mathrm{Ind}_{B}^{G_rB} (\mathbb{k}_{(p^r-1)\rho} \otimes (Fr_B^*)^r M)$
$\simeq R^i \mathrm{Ind}_B^G (\mathbb{k}_{(p^r-1)\rho} \otimes (Fr_B^*)^r M)$.

③ & ④ complete the proof. $\square$.

How does Kempf's vanishing Theorem help us understand characters of $M(\lambda)$ and $W(\lambda)$?

Cartan involution

① $ch(W(\lambda)) = ch(H^0(\lambda))$
(Recall that $\exists\ \tau \in Aut(G)$ s.t. $\tau^2 = id$, $\tau|_T = id_T$, and $\tau(U_\alpha) = U_{-\alpha}\ \forall \alpha \in \Phi$. Then $W(\lambda) \simeq {}^{\tau}H^0(\lambda)$.)

② $ch(H^0(\lambda)) = \chi(\mathbb{k}_\lambda) = \sum_{i\geq 0} (-1)^i ch\, H^i(\mathbb{k}_\lambda)$
This is by Kempf vanishing Thm.

§3.2 Ext Vanishing and Highest W.t Structure

Prop $\mathrm{Ext}^i_G(W(\lambda), M(\mu)) = 0\ \forall i>0$, $\lambda, \mu \in \Lambda^+$
Proof Note that $(\cdot)^*$ is a contravariant self-equiv. on $Rep(G)$.
Then
$\mathrm{Ext}^i(W(\lambda), M(\mu)) = \mathrm{Ext}^i(W(\mu^*), M(\lambda^*))$. Assume that $\lambda \not< \mu$.

$\mathrm{Ext}^i_G(W(\lambda), M(\mu)) = \mathrm{Ext}^i_B(W(\lambda), \mathbb{k}_{-\mu^*})$.

All T-w.ts in $W(\lambda) \geq -\lambda^*$
All T-w.ts in $\mathbb{k}[B]$ are $\geq 0$ linear combination of positive roots
$\Rightarrow$ all wts in an inj. resolution are $\leq -\mu^*$ w/ $\mu^*$ only in deg 0.

$\Rightarrow \mathrm{Ext}^i_B(W(\lambda), \mathbb{F}_{-w^*}) = 0$.

$\square$

<u>Exercise</u> $\mathrm{Rep}_{\leq\mu}(G)$ is a highest wt. cat. w/ simples $L(\lambda)$, standard, ~~$H(\lambda)$~~ $W(\lambda)$, and costandards $M(\lambda)$ for $\lambda \leq \mu$.

① $\mathrm{Ext}^i_\mu(W(\lambda), M(\mu)) \simeq H^i(G, H^0(-w_0\lambda) \otimes M(\mu))$.

② $H^i(G, H^0(-w_0\lambda) \otimes H^0(\mu)) \simeq H^i(B, H^0(-w_0\lambda) \otimes \mathbb{F}_\mu)$
$\simeq H^i(B, \mathbb{F}_{-w_0\lambda} \otimes H^0(\mu))$. $(*)$

$\uparrow$

Tensor identity & Kempf's Vanishing

LHS $(*) \neq 0$ if $\exists$ w.t. $\nu$ of $H^0(-w_0\lambda)$ and $\nu'$ of $H^0(\mu)$ s.t.
$\mu + \nu, -w_0\lambda + \nu' \in -\mathbb{Z}_{\geq 0}\Phi^+$ and
$ht(\mu+\nu)$, $ht(-w_0\lambda + \nu') \leq -i$.
$ht(\sum n_\alpha \alpha) = \sum n_\alpha$.
$\alpha$ simple

$0 \to \mathbb{k} \to I_1 \to I_2 \to \cdots$ inj. resolu of $\mathbb{k}$ as a $B$-mod.
$I_0$: inj hull of $\mathbb{k}$, all wt. space of $I_j$ f.d.

| | smallest wt. |
|---|---|
| $H^0(-w_0\lambda)$ | $-\lambda$ |
| $H^0(\mu)$ | $w_0\mu$ |

all wt. $\nu$ of $I_j$,
$\nu \in \mathbb{N}\Phi^+$, $ht(\nu) \geq j$.

$\Rightarrow -w_0\lambda + w_0\mu = 0 \Rightarrow \lambda = \mu$.
$0 = ht(-w_0\lambda + w_0\mu) < ht(-w_0\lambda + \nu) \leq -i$. $\Rightarrow \Leftarrow$ $i > 0$.

## §4. Linkage Principle.

**Question**

Which $L(\lambda)$ occurs w/ nonzero multiplicity in ~~non-zero many times~~ in $M(\lambda)$ and $W(\lambda)$? (In fact $W(\lambda)$ and $M(\lambda)$ have the same characters $\Rightarrow$ belong to the same $K_0$-class).

**Recall**

$\mathfrak{g}$: complex semisimple Lie algebra. $\mathcal{O}$: category $\mathcal{O}$

simple objects are parametrized by $\lambda \in \mathfrak{h}^*$, $\mathfrak{h} \subset \mathfrak{g}$ Cartan.

Bruhat order $\preceq$ on $\Lambda$: $\mu \preceq \lambda$ if $\exists\ \lambda_0 = \lambda, \lambda_1, \ldots, \lambda_k = \mu$ s.t. $\lambda_{i+1} < \lambda_i$ and $\lambda_{i+1} = s_{\beta_i} \cdot \lambda_i$ for some root $\beta_i$. Then

$[\nabla(\lambda) : L(\mu)] \neq 0 \Rightarrow \mu \preceq \lambda$.

### §4.1 Main Result.

- Dual affine Weyl group $W^a := W \ltimes \Lambda_r$, $\Lambda_r$: root lattice
- $w \cdot_p \lambda$ as usual $w \in W$.
- $t_\nu \cdot_p \lambda := \lambda + p\nu$ $\forall\ \nu \in \Lambda_r$
- " $\cdot_p$ ": $p$-rescaled dot-action
- $A^+ = \{\lambda \in \Lambda \mid \langle \lambda + \rho, \alpha_i^\vee \rangle \geq 0\ \forall\ i = 1, 2, \ldots, r,\ \langle \lambda + \rho, \alpha_0^\vee \rangle \geq -p\}$.

For $SL_2$, $A^+ = \{-1, 0, \ldots, 4\}$

- Def (Linkage order) $\lambda, \mu \in \Lambda$

$\lambda \uparrow \mu$ if $\exists\ \lambda_0 = \mu, \lambda_1, \ldots, \lambda_k = \lambda \in \Lambda$ and affine reflections $s_0, \ldots, s_{k-1}$ s.t. $\lambda_i = s_{i-1} \cdot_p \lambda_{i-1}$ and $\lambda_i \leq \lambda_{i+1}$ for $i = 1, 2, \ldots, k$.

- Remark $\lambda \uparrow \mu \Rightarrow \lambda \leq \mu$ and $\mu \in W^a \cdot_p \lambda$.

### §4.2 Block Decomposition

Coro If $\mathrm{Ext}^1(L(\lambda), L(\mu)) \neq 0$, then $\mu \in W^a \cdot_p \lambda$

$\forall\ \xi \in A^+$ $\mathrm{Rep}_\xi(G)$: the Serre span of $L(\lambda)$, $\lambda \in W^a \cdot_p \xi$.

Then $\mathrm{Rep}(G) = \bigoplus_{\xi \in A^+} \mathrm{Rep}_\xi(G)$.

$G = SL_2$

$\Lambda = \mathbb{Z}x_1$, $x_1 : \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \mapsto t$

$\Lambda_r = 2\mathbb{Z}x_1$.

$W = \{1, s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\}$. $\forall\, nx_1 \in \Lambda$, $s(nx_1) = -nx_1$.

For $\lambda = nx_1 \in \Lambda$, and $\nu = 2mx_1 \in \Lambda_r$

$$(s, t_{2mx_1}) \cdot_p nx_1 = s(nx_1 + 2mpx_1 + x_1) - x_1$$
$$= -(n + 2mp + 2)x_1.$$

So, if $nx_1 \uparrow (s, t_{2mx_1}) \cdot_p nx_1$, then $-mp \geq n+1$.

$$(1, t_{2mx_1}) \cdot_p nx_1 = nx_1 + 2mpx_1$$

So if $nx_1 \uparrow (1, t_{2mx_1}) \cdot_p nx_1$, then $mp \geq 0$.

$$(s, t_{2m_2x_1}) \cdot_p (-n - 2mp - 2)x_1$$
$$= s((-n - 2mp - 2)x_1 + 2m_2px_1 + x_1) - x_1$$
$$= s((-n - 2mp + 2m_2p - 1)x_1) - x_1$$
$$= (n + 2mp - 2m_2p)x_1.$$

For $p = 5$.